A Study of Renormalization Group Methods Applied to Fluid Turbulence

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Abstract

This thesis presents a renormalization group method to tackle the problem of reducing the number of degrees of freedom necessary to describe fluid turbulence.

Attempts to apply renormalization group methods to fluid turbulence have been in general based on an existing formulation of a multivariate normal model of turbulence. This model is an appropriate zero-order model for perturbation theories of turbulence in the context of the moment-closure problem because of the Gaussian distribution of the zero-order field for which moments of all orders can expressed in terms of pair correlations. However, this is not a relevant attribute for renormalization group methods. A critical review of renormalization group methods based on the multivariate normal model is presented.

An alternative approach was developed by McComb and Watt [Phys. Rev. A, **46**, 4797 (1992)], who introduced a formal conditional average. The rescaling two-field theory leads to the derivation of a recursion relation, which eliminates finite blocks of turbulent velocity modes while maintaining the form invariance of the dynamical equation. In the present thesis, it is shown that the theory was heuristic in some respects and there were inadequate explanations in the procedure of that theory.

A new formulation of renormalization group method, based on an alternative interpretation of the two-field theory, is presented here. A new zero-order model field is proposed for the Fourier modes in which there is no coupling between the high-wavenumber band of modes being eliminated and the remaining low-wavenumber modes. This model has the property that high-wavenumber modes can be eliminated without the need for a conditional average. The model field is then made the basis of a formal perturbation series which recovers the results of the two-field theory in a way which eliminates certain ambiguities, and allows one to see clear relationship between a turbulent velocity field and the zero-order model field. The results are the systematic derivations of an equation for the high-wavenumber modes that exhibits form invariance under the renormalization-group transformation, and an expression for the effective viscosity for use in the computational simulation of homogeneous and isotropic turbulence. A value for the Kolmogorov constant of $\alpha = 1.6$ is obtained, when the fixed point for the effective viscosity is numerically calculated.

The method is then extended to the problem of a passive scalar field convected by a turbulent velocity. A double expansion of the passive scalar and the turbulent velocity fields is applied for this case. The results are similar to those for the turbulent velocity field: the form invariance of the equation of motion under the renormalization-group transformation and the recursive form of an effective diffusivity for the passive scalar field.

Finally, the method of Martin, Siggia and Rose [Phys. Rev. A, **8**, 423 (1973)], in which the quantum-statistical perturbation theory for classical statistical dynamics was developed, is reviewed, and its application to the problem of the turbulent velocity field is briefly presented.

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Chapter 1

Introduction

1.1 Homogeneous and isotropic turbulence

Turbulence is a general phenomenon observed in a wide variety of flows: the smoke rising from a cigar, the mixing cream in a coffee cup, the rapid flow of a river passing an obstacle and even the formation of galaxies in early universe exhibit the disordered behavior of turbulent motion.

In 1883, the phenomenon of turbulence was first studied by Reynolds [1]. He observed streams of dye in water flow through long straight pipes of constant diameter. In his experiment, it was found that streams of dye would remain orderly (i.e. laminar state) for the velocity of water flow below a certain critical speed. Above a critical speed, however, streams of dye no longer followed a characteristic of laminar flow, but broke up abruptly at some distance from the pipe entrance. This was the first experimental work to visualize turbulent flow. Based on his experimental work, Reynolds introduced a dimensionless parameter that is useful to characterize fluid flow,

$$Re = \frac{UL}{\nu}, \tag{1.1}$$

where ν is the kinematic viscosity of fluid flows, and U and L are characteristic velocity and length scales (e.g. the mean velocity of fluid flows and the diameter of the pipe in the case of flow through a pipe). He found that the transition from

stable laminar to unstable turbulent flow could occur above a certain value of the Reynolds number ($Re \sim 2000$ for his case).

In general, it is the fact that, where the kinematic viscosity ν is sufficiently low (i.e. the Reynolds number is sufficiently high), the motion of fluid flows is extremely unstable, and the velocity of fluid flows takes approximately a random value in time and space. In other words, if the Reynolds number is high enough, the essential features of turbulence will be unpredictability and randomness. Under these circumstances, it is possible to assume that such a flow is a *homogeneous* turbulence, where average properties are invariant under translation of axes. The restriction to (spatial) homogeneity simplifies the theoretical treatment of turbulence enormously. Furthermore, an assumption about direction symmetry of average properties (i.e. *isotropic* turbulence) can also be made; in order to simplify the disordered behavior of turbulent motion, and thus make it more tractable in theoretical respects. Although homogeneous and isotropic turbulence is not an in reality existing type of flow, it is possible to get a close approximation to homogeneous and isotropic turbulence in laboratory for theoretical predictions to be examined. A common way of producing this type of flow is by the use of a grid in a wind or water tunnel (so-called grid turbulence). Once the fluid is led through a regular grid, over regions somewhere greater than its characteristic length scale, the turbulence is approximately homogeneous and isotropic.

While there is no unique and rigorous mathematical model that explains all the different situations of turbulent motion, it is in general recognized that the simplest non-trivial model is that of homogeneous and isotropic turbulence in an incompressible fluid, and a great deal of theoretical work has been built on this foundation; reference [2] extensively summarizes the early work in this field. This type of turbulence is the subject of this thesis, which allows us to analyze the physical phenomenon of turbulence in terms of analytical statistical formulations. This will be considered further in the following sections.

1.2 Statement and formulation of the problem

1.2.1 The equation of motion

We now consider the governing equation for the motion of a turbulent fluid:

$$\frac{\partial}{\partial t}u_\alpha(\boldsymbol{x},t) + u_\beta(\boldsymbol{x},t)\frac{\partial}{\partial x_\beta}u_\alpha(\boldsymbol{x},t) = -\frac{1}{\rho}\frac{\partial}{\partial x_\alpha}\mathcal{P}(\boldsymbol{x},t) + \nu_0\frac{\partial^2}{\partial x_\beta \partial x_\beta}u_\alpha(\boldsymbol{x},t), \quad (1.2)$$

where $u_\alpha(\boldsymbol{x},t)$ is the fluid velocity, ρ is the density of the fluid, $\mathcal{P}(\boldsymbol{x},t)$ is the pressure in the fluid, and ν_0 is the molecular kinematic viscosity. The Einstein summation is assumed for duplicated indices. Equation (1.2) is known as the Navier-Stokes equation expressing the conservation of momentum of the fluid, and *subject to* initial and boundary conditions.

For the continuity equation expressing the conservation of mass of the fluid, we shall only give our attention to the case of the fluid, where the incompressibility assumption (i.e. the density of the fluid is uniform) is effectively valid [2]. Then, the equation of continuity has the form

$$\frac{\partial}{\partial x_\alpha}u_\alpha(\boldsymbol{x},t) = 0. \quad (1.3)$$

The pressure, $\mathcal{P}(\boldsymbol{x},t)$, in equation (1.2), can be eliminated using the equation of continuity to give the solenoidal Navier-Stokes in x-space [3]:

$$\left[\frac{\partial}{\partial t} - \nu_0\boldsymbol{\nabla}^2\right]u_\alpha(\boldsymbol{x},t) = M_{\alpha\beta\gamma}\{\boldsymbol{\nabla}\}u_\beta(\boldsymbol{x},t)u_\gamma(\boldsymbol{x},t). \quad (1.4)$$

Here, we assume the boundary condition that $\boldsymbol{u}$ goes to zero as $\boldsymbol{x}$ goes to infinity, and introduce the symmetric operator $M_{\alpha\beta\gamma}\{\boldsymbol{\nabla}\}$ such that

$$M_{\alpha\beta\gamma}\{\boldsymbol{\nabla}\} = -\frac{1}{2}\left[\frac{\partial}{\partial x_\beta}D_{\alpha\gamma}\{\boldsymbol{\nabla}\} + \frac{\partial}{\partial x_\gamma}D_{\alpha\beta}\{\boldsymbol{\nabla}\}\right], \quad (1.5)$$

and the projection operator $D_{\alpha\beta}(\boldsymbol{\nabla})$ as

$$D_{\alpha\gamma}\{\boldsymbol{\nabla}\} = \delta_{\alpha\beta} - \frac{\partial^2}{\partial x_\beta \partial x_\beta}\int_V d^3x'\, G(\boldsymbol{x},\boldsymbol{x}'), \quad (1.6)$$

where $G(\boldsymbol{x}, \boldsymbol{x}')$ is the Green's function which satisfies Laplace's equation in the form

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{1.7}$$

subject to the condition

$$n_\alpha \frac{\partial}{\partial x_\alpha} G(\boldsymbol{x}, \boldsymbol{x}') = 0, \tag{1.8}$$

and $\boldsymbol{n}$ is the unit inward normal vector to the surface at $\boldsymbol{x}$.

Theoretical approaches for fluid turbulence in general begin by introducing the Fourier analysis for the equation (1.4). This leads to simplification by converting differential operators into multipliers, and gives us a relatively simple picture of the physics of turbulence.

We begin by introducing the integral Fourier transform of the velocity field $u_\alpha(\boldsymbol{x}, t)$ which can be expressed as follows:

$$u_\alpha(\boldsymbol{x}, t) = \int_{k<\Lambda} d^3k \; u_\alpha(\boldsymbol{k}, t) \exp(i\boldsymbol{k}\cdot\boldsymbol{x}), \tag{1.9}$$

where Λ is a cut-off wave-number. Then, the solenoidal Navier-Stokes equation, (1.4), becomes:

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha(\boldsymbol{k}, t) = M_{\alpha\beta\gamma}(\boldsymbol{k}) \int_{j<\Lambda} d^3j \; u_\beta(\boldsymbol{j}, t) u_\gamma(\boldsymbol{k} - \boldsymbol{j}, t), \tag{1.10}$$

where the inertial transfer operator $M_{\alpha\beta\gamma}(\boldsymbol{k})$ is given by

$$M_{\alpha\beta\gamma}(\boldsymbol{k}) = \frac{1}{2i}\Big[k_\beta D_{\alpha\gamma}(\boldsymbol{k}) + k_\gamma D_{\alpha\beta}(\boldsymbol{k})\Big], \tag{1.11}$$

and the projector $D_{\alpha\beta}(\boldsymbol{k})$ is expressed in terms of the Kronecker delta $\delta_{\alpha\beta}$ as

$$D_{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{|\boldsymbol{k}|^2}, \tag{1.12}$$

which arises due to the incompressibility condition.

As Fourier analysis has been introduced for the study of turbulence, it now becomes an example of a many-body problem with many degrees of freedom (i.e. the velocity modes in wave-number space) as encountered in statistical physics

and the study of critical phenomena. The convolution form of the velocity modes in the RHS of equation (1.10) gives the fact that each mode is, in principle, dependent on every other one, and this is the *non-linearity* of the Navier-Stokes equation.

1.2.2 Stirred Navier-Stokes equation

The theoretical study of homogeneous and isotropic turbulence has the strict status of a *gedanken* experiment, and requires some input of energy to compensate for the losses due to the viscous dissipation. Note that the solenoidal Navier-Stokes equation (1.10) contains no input of energy, and thus it cannot describe stationary turbulence. The theoretical model for this situation is therefore a stationary field driven by a hypothetical stirring force $\boldsymbol{f}$, which has been added into the RHS of equation (1.10) to sustain the turbulent state, such as

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha(\boldsymbol{k},t) = M_{\alpha\beta\gamma}(\boldsymbol{k}) \int d^3j \; u_\beta(\boldsymbol{j},t) u_\gamma(\boldsymbol{k}-\boldsymbol{j},t) + f_\alpha(\boldsymbol{k},t), \quad (1.13)$$

where $\boldsymbol{f}$ should be solenoidal $[k_\alpha f_\alpha(\boldsymbol{k},t) = 0]$, in order to maintain the incompressibility of the velocity field.

For many years, attempts to solve the problem of fluid turbulence have been based on taking the probability distribution of the forces $\boldsymbol{f}$ to be multivariate normal (or Gaussian) with their correlation function to be given by

$$\langle f_\alpha(\boldsymbol{k},t) f_\beta(\boldsymbol{k}',t')\rangle = W(k) D_{\alpha\beta}(\boldsymbol{k}) \delta(\boldsymbol{k}+\boldsymbol{k}') \delta(t-t'), \quad (1.14)$$

where $W(k)$ remains to be specified. The advantages of these prescriptions for the stirring forces will become apparent later, and we will referred to this class of model as the *multivariate normal model*, hereafter.

It should be mentioned here that a method developed in this thesis is *not* based on the multivariate normal model, and a different specification of stirring forces shall be given in section 3.2.

1.2.3 Energy picture: the Kolmogorov hypothesis

Now, consider the second moment of the turbulent velocity field, which is related to the energy spectrum of the system. For homogeneous, isotropic and stationary turbulence, the second moment can be expressed as follows:

$$\langle u_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{k}',t')\rangle = Q(k,t-t')D_{\alpha\beta}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}'), \tag{1.15}$$

where $Q(k,t-t')$ is the spectral density, and the ensemble average, over all possible realizations of the Navier-Stokes equation, is denoted by the Dirac bracket $\langle\cdots\rangle$. Then, we may define an energy spectrum $E(k,t)$ by

$$E(k,t) = 4\pi^2 Q(k,t), \tag{1.16}$$

and the total kinetic energy per unit mass $E(t)$ is the integral of the spectrum over all wave-numbers:

$$E(t) = \int_0^\infty dk\ E(k,t). \tag{1.17}$$

Now, in order to obtain the equation for $E(k,t)$, we take the following steps:

(i) multiply each term in equation (1.13) by $u_\delta(-\boldsymbol{k},t)$;

(ii) rewrite equation (1.13) for $u_\delta(-\boldsymbol{k},t)$, and multiply it by $u_\alpha(\boldsymbol{k},t)$;

(iii) add the equations produced by steps (i) and (ii) and average, then set $\alpha = \delta$ and sum over indices.

This procedure yields

$$\frac{d}{dt}E(k,t) + 2\nu_0 k^2 E(k,t) = T(k,t) + W(k,t), \tag{1.18}$$

where $W(k,t)$ represents the work done by the stirring forces $\boldsymbol{f}$ on the turbulent velocity field $\boldsymbol{u}$, and $T(k,t)$ represents the complex non-linear energy transfer term, which it can be shown conserves energy globally [3], or

$$\int_0^\infty dk\ T(k,t) = 0. \tag{1.19}$$

If we integrate each term in equation (1.18), then from equations (1.17) and (1.19) we have the energy balance equation

$$\frac{d}{dt}E(t) + \int_0^\infty dk\ 2\nu_0 k^2 E(k,t) = W(k,t). \tag{1.20}$$

The typical interpretation of equation (1.20) is that energy in the system at low wave-number by $W(k,t)$ is transferred to high wave-number, where it is dissipated by the viscous term. For the stationary turbulence, where the input of energy balances the output of energy, the rate of energy dissipation per unit mass per unit time of fluid ε can be defined as

$$\varepsilon \equiv -\frac{d}{dt}E(t) = \int_0^\infty dk\ 2\nu_0 k^2 E(k), \tag{1.21}$$

where $E(k) \equiv E(k,0)$ defined by equation (1.16).

In 1941, Kolmogorov [4, 5] postulated the idea of universality based on the concept of the *inertial range*. According to his picture [6], for very large Reynolds numbers, the velocity field in wave-number space can be divided into three ranges (see also figure 1.1):

- At low wave-number $k \sim L^{-1}$, there is the energy-containing scale of the macroscopic size where the energy is injected due to the external forcing or instabilities. The scale L is a characteristic macroscopic length associated to the largest relevant dimension of the turbulent system;

- For wave-number $k \sim k_d$, the small eddies are dissipated via inter-molecular collisions, where k_d is the Kolmogorov dissipation wave-number which has the the form [1]

$$k_d = \left(\frac{\varepsilon}{\nu_0^3}\right)^{1/4}. \tag{1.22}$$

  Here, ε is the rate of energy dissipation per unit mass per unit time of fluid, and ν_0 is the kinematic viscosity;

[1] In his two seminal papers of 1941 [4, 5], Kolmogorov proposed that the only relevant parameters in the inertial interval are the dissipation rate, ε, and the kinematic viscosity, ν. Thus, on dimensional grounds, we can have a characteristic inverse length scale, equation (1.22)

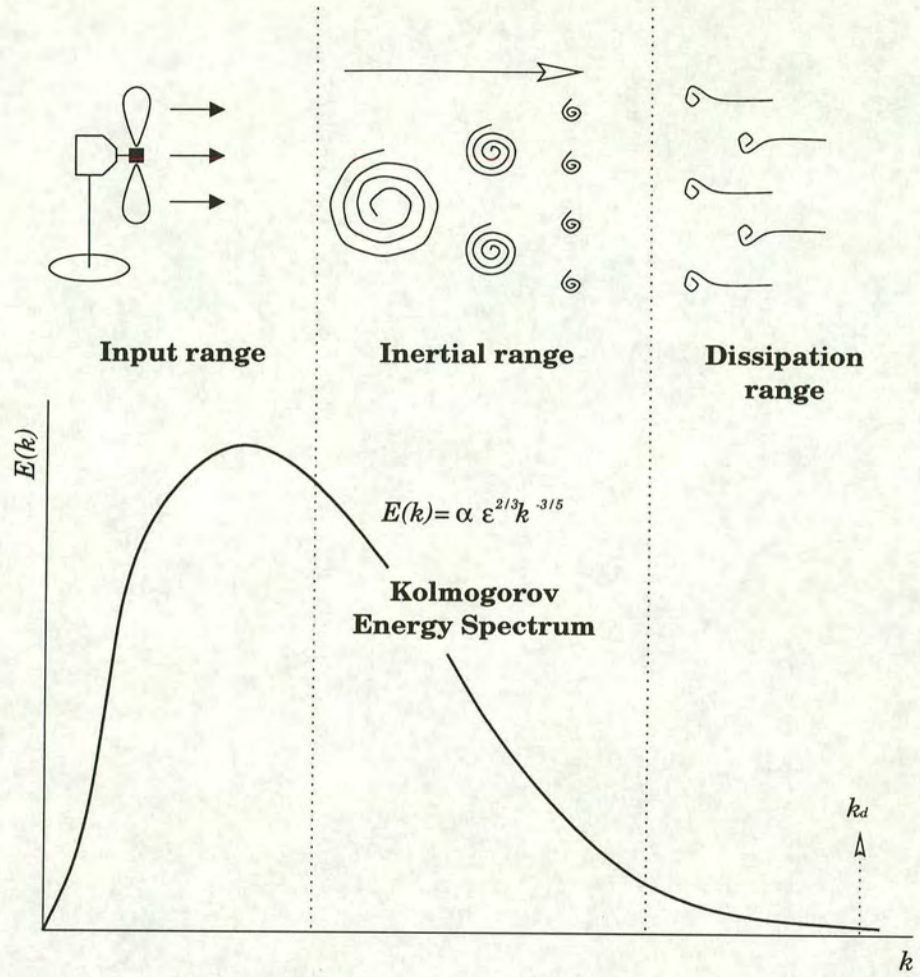

Figure 1.1: A schematic illustration of the Kolmogorov picture with three ranges: input, inertial and dissipation.

- In between the previous two ranges, far from the both ends, the *inertial range* exists where the energy is transferred from larger eddies to smaller eddies via the local interactions between eddies of similar size. In the inertial range, the energy spectrum may take a universal form and, on dimensional grounds, can be written as

$$E(k) = \alpha \varepsilon^{2/3} k^{-5/3}, \tag{1.23}$$

where α is so-called the Kolmogorov constant.

Equation (1.23) is known as the Kolmogorov energy spectrum, and the calculation of the Kolmogorov constant has been the goal of fundamental theories. The

Kolmogorov energy spectrum has been confirmed by experiment [7], although the value of the Kolmogorov constant remains open to question. An extensive survey of the experimental literature on the Kolmogorov constant was accomplished by Monin and Yaglom [8], and they concluded that the accepted range for the Kolmogorov constant is $\alpha = 1.2 - 1.8$. Recently, Sreenivasan [9] carried out the survey of experimental and computational work, and the result showed that $\alpha = 1.619$ was the best estimate with an error of 3.4%.

However, it is often asked whether the Kolmogorov hypothesis really does represent the energy picture of turbulence. Shortly after two papers of Kolmogorov [4, 5], Landau and Lifshitz [10] pointed out the problem, the so-called *intermittency* phenomenon, essentially due to the variation of the rate of energy dissipation:

$$\varepsilon(\boldsymbol{x}, t) = \frac{1}{2} \sum_{\alpha} \sum_{\beta} \left[\frac{\partial u_{\alpha}(\boldsymbol{x}, t)}{x_{\beta}} + \frac{\partial u_{\beta}(\boldsymbol{x}, t)}{x_{\alpha}} \right]^2 , \tag{1.24}$$

whereas Kolmogorov [4, 5] considered as a mean value. Thus it has been argued that the distribution of the dissipation rate must be taken into account for the derivation of the Kolmogorov energy spectrum. Since then, many experiments [7, 11, 12] have been carried out to confirm the existence of intermittency in turbulence, and numerous theories [13, 14, 15] including Kolmogorov [16] himself have also been presented to explain the phenomenon. On the other hand, it has also been argued that the derivation of the Kolmogorov result[4] for second-order structure function can be exact [17, 18] and recently [15, 19, 20].

Therefore, it is difficult to reach any conclusion here for the question: whether the Kolmogorov hypothesis really does represent turbulence or not. Instead, in chapter 4, we will attempt to evaluate a value for the Kolmogorov constant from the results of our theory, by assuming that energy spectrum is a power law which leads to the Kolmogorov energy spectrum described in equation (1.23).

1.2.4 Developments in the study of turbulence

As previously discussed, turbulence is the archetypal non-linear phenomenon. Despite efforts to simplify the phenomena, no-one has yet achieved the full analytical solution of the turbulent velocity field at all points in time and space, and indeed the topic is often referred to as the last unsolved problem of classical physics. Therefore, any approach to the study of such systems can only be of statistical nature, that is to solve the equations of motion for mean quantities.

For homogeneous and isotropic turbulence, the mean velocity field is zero. Thus, the simplest non-trivial quantity to calculate is the second moment of the velocity field $\langle \boldsymbol{uu} \rangle$, which can be formed by multiplying each term of equation (1.13) by $u_\delta(\boldsymbol{l}, t')$ and averaging[2]:

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] \langle u_\alpha(\boldsymbol{k}, t) u_\delta(\boldsymbol{l}, t') \rangle = M_{\alpha\beta\gamma}(\boldsymbol{k}) \int d^3 j \, \langle u_\beta(\boldsymbol{j}, t) u_\gamma(\boldsymbol{k} - \boldsymbol{j}, t) u_\delta(\boldsymbol{l}, t') \rangle. \tag{1.25}$$

Now, we are faced with the problem of closing the moment hierarchy, because the equation for second moment $\langle \boldsymbol{uu} \rangle$ contains the third moment $\langle \boldsymbol{uuu} \rangle$, and the equation for $\langle \boldsymbol{uuu} \rangle$ contains $\langle \boldsymbol{uuuu} \rangle$ and so on. This problem is usually referred to as the *closure* problem. Consequently, this is the main problem in the study of turbulence to find appropriate method of converting the infinite moment hierarchy into a closed set.

The attempts to solve the problem could be categorized into three areas: ad-hoc closures; renormalized perturbation theories; renormalization group theories. In the ad-hoc closures, the postulations to truncate the moment hierarchy have been usually made: the closure approximations employing the Heisenberg effective viscosity model [21] and the quasi-normality hypothesis [22, 23, 24]. More general discussions about the ad-hoc closures can be found in references [3], [25] and [26].

On the other side, the attempts to renormalize theories have been based on the

[2] Here we do not consider the term $\langle \boldsymbol{fu} \rangle$ because of insufficient information about $\boldsymbol{f}$ at this stage.

Navier-Stokes equation with a specific stirring force:

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha(\boldsymbol{k},t) = f_\alpha(\boldsymbol{k},t) + \lambda M_{\alpha\beta\gamma}(\boldsymbol{k}) \int d^3j \; u_\beta(\boldsymbol{j},t) u_\gamma(\boldsymbol{k}-\boldsymbol{j},t), \quad (1.26)$$

where a bookkeeping parameter λ has been placed in front of the nonlinear term.

Evidently, the random force, $\boldsymbol{f}$, has to satisfy the equation of continuity and, in order to make it appropriate for perturbation theory, it is necessary to choose it to have a distribution that is multivariate normal and highly uncorrelated in time. Then, this model gives the simplest possible way of describing the system by extracting the non-linear coupling term at the zero-order field, such that

$$u_\alpha(\boldsymbol{k},t) = u_\alpha^{(0)}(\boldsymbol{k},t) + \lambda u_\alpha^{(1)}(\boldsymbol{k},t) + \lambda^2 u_\alpha^{(2)}(\boldsymbol{k},t) + \cdots, \quad (1.27)$$

in terms of the zero-order field $\boldsymbol{u}^{(0)}$ as

$$u_\alpha^{(0)}(\boldsymbol{k},t) = D_{\alpha\beta}(\boldsymbol{k}) \int_{-\infty}^{t} dt' \; G^{(0)}(\boldsymbol{k};t,t') f_\beta(\boldsymbol{k},t'), \quad (1.28)$$

where $G^{(0)}(\boldsymbol{k};t,t')$ is the Green's function representing the purely viscous response of the fluid, and which satisfies

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] G^{(0)}(\boldsymbol{k};t,t') = \delta(t-t'). \quad (1.29)$$

These set-ups are the starting point of theories based on the multivariate normal model to find a way of summing certain classes of the perturbation series to all orders, having used the Gaussian properties of the zero-order field to evaluate the moments of all orders.

The application of renormalized perturbation theories to the closure problem in the study of fluid turbulence was pioneered by Kraichnan [27], who introduced the well-known *direct interaction approximation* (or DIA). The procedures of the DIA can be briefly summarized as follows:

- treat the non-linear term in equation (1.26) as a small disturbance;
- expand the velocity field and the response function as perturbation series, having used the multivariate normal model [see equations (1.27) - (1.29)];
- sum certain classes of terms to all orders to produce a renormalized perturbation series;
- truncate the renormalized expansion at the lowest-order term, where the solution is non-trivial or the average properties of field variables do not vanish.

As a result, Kraichnan derived closed equations for the correlation and response functions. Later, having introduced a relatively simple model, he also showed that the result of the DIA could be valid, where the non-linear terms are not essentially small [28]. The obvious drawback of his method might be, however, that the DIA in the study of turbulence predicted a $-3/2$ power law for the energy spectrum in the inertial range, whereas the $-5/3$ Kolmogorov energy spectrum has been predominant in the study of turbulence (see section 1.2.3).

Other type of renormalized perturbation theories, followed after the original formulation of DIA, have been developed: Edward-Fokker-Plank method [29]; Lagrangian-history DIA [30, 31]; self-consistent field theory [32]; local energy-transfer theory [33]. Each of these methods has its own rights and difficulties, and they are well documented in reference [3]. In addition, the common difficulty in the application of renormalized perturbation theories to the Navier-Stokes equation might be that their procedures involve a truncation of uncertain convergence properties. It has been pointed out by Kraichnan [34] that the analytical properties of renormalized series in any approximation based on a truncation are almost unknown, and consequently such a truncation might generate an artificial effect on energy transfer of the given system. For example, the DIA does not have the Kolmogorov inertial-range spectrum as its solution.

In addition to the discussion of renormalized perturbation theories, an interesting development was made by Martin, Siggia and Rose [35], who introduced an ingenious way of overcoming the closure problem in the study of a classical dynamic system (see appendix A for more discussions about reference [35]). They developed a method parallel to the mathematical formalism of quantum field theory and the statistical dynamics of a classical system; and their method has been extended to critical dynamics by DeDominicis and Peliti [36], and to electromagnetic problem by Krommes and Kleva [37]. Nevertheless, the application of their method to the study of the Navier-Stokes equation has been restricted to the lowest-order of renormalized perturbation theory or DIA.

Recently, iterative algorithms like the renormalization group have been investigated to reduce the number of degrees of freedom which exist in the study of the solenoidal Navier-Stokes equation, and this will be the subject of this thesis.

1.3 Thesis overview

The work given in this thesis is a development of renormalization group approaches to the problem of fluid turbulence.

As the renormalization group method has succeeded in the derivation of universal scaling relations for systems at the critical point [38, 39], it has long been thought desirable to take over the renormalization group method and apply it to turbulence. However, it has not proven easy in practice due to the strong coupling that exists between all the modes of the turbulent velocity field, and the situation is still not resolved after the first attempt by Forster, Nelson and Stephen [40, 41]. Attempts to use renormalization group methods to apply turbulence can be in general divided into two classifications: one is field-theoretic renormalization group method and the other is recursive-type renormalization group method.

The field-theoretic renormalization group method was pioneered by Forster, Nelson and Stephen [40, 41], and studied by many others [42, 43, 44, 45, 46, 47]. The method is based on the formulation of a multivariate normal model of turbulence described in section 1.2. This model is an appropriate zero-order model for perturbation theories of turbulence in the context of the moment-closure problem because it gives the simplest non-trivial statistical way of describing the system. The basic property which makes this model soluble is the Gaussian properties of the zero-order field for which moments of all orders can expressed in terms of pair correlations. However, it is not obvious that this is a relevant property for renormalization group approaches.

This class of models based on the work of Forster, Nelson and Stephen is referred to as *stirred hydrodynamics.* Recently, the concept of this formalism was extended to real turbulence by Yakhot and Orszag. They proposed the *correspondence principle*, which states, if the stirring forces are chosen to give the same energy spectrum as in a real turbulence, then the associated numerics will also be correct for that flow. However, it remains to be seen whether or not some more fundamental justification can be found.

In contrast, the recursive-type renormalization group method was introduced by Rose [48], who studied the passive convection of a scalar containment and by McComb [49] for the case of the Navier-Stokes equation. In the latter case, the method is based on the derivation of a recurrence relation, which eliminates finite blocks of modes while maintaining the form invariance of the dynamical equation. On the other hand, the method of Rose was also extended directly to the velocity field by Zhou, Vahala and Hossain [50], who suggest a way of maintaining the form invariance of the new equation that includes the triple non-linear term.

The difference between the two approaches can be summarized as follows; in the field-theoretic renormalization group method, one ensures that the non-linear coupling is small by performing calculations in a low wave-number region, where

the viscous and stirring term are in balance, whereas in the second method, calculations are executed in a high wave-number region, where the viscous effects are dominant and the non-linear coupling is inherently small. Recently, this type of renormalization group method was developed as the *two-field theory* by McComb and Watt [51]; in particular a rigorously formulated conditional average was introduced to deal with the non-linear coupling. However, the theory was heuristic in some respects, and we have found that there were inadequate explanations in the procedure of that theory.

In this thesis, we present an alternative and systematic derivation of the calculations previously given in the two-field theory, by means of a perturbation expansion. In order to achieve this we introduce a new model of turbulence. This model is an appropriate zero-order model for perturbative implementation of renormalization group theories of turbulence because it provides the possibility of describing the high wave-number and the low wave-number modes of the system independently, by suppressing the phase information between them. The basic property which makes this model soluble is that the conditional average can be evaluated exactly for the model field. The model is then made the basis of a formal perturbation series which re-derives the results of the two-field theory in a way which eliminates certain ambiguities and allows one to see a clear relationship between a turbulent velocity field and the zero-order model field.

The thesis is laid out as follows. In chapter 2, renormalization group methods applied to the study of fluid turbulence are discussed. Then, the motivation for the work given in this thesis would appear as the problems found in other methods are examined. In chapter 3, we introduce a new approach to the problem, which looks for a systematic re-derivation of the previous results of the two-field theory [51]. A model is proposed, in which there is no coupling between the eliminated and the remaining modes. A formal perturbation by setting up our model as a zero-order field is then employed. The results are systematic derivations of an equation for the remaining velocity modes, and an expression for the effective

viscosity contains the effect of the eliminated velocity modes. Chapter 4 is an analysis of the theory given in chapter 3. It is shown that a value of the Kolmogorov constant can be obtained, when the results presented in chapter 3 are numerically calculated. We then examine an approximation made in our theory, and discuss about triple non-linearities which exist in the procedure of renormalization group method for the Navier-Stokes equation. Chapter 5 is an extension of the work given in chapter 3 to the case of a passive scalar field convected by the turbulent velocity field. In chapter 6, we summarize our work and suggest future work. Finally, in appendix A, the method of Martin, Siggia and Rose and its application to the problem of the turbulent velocity is briefly presented.

Chapter 2

Renormalization Group Methods Applied to Fluid Turbulence

2.1 Introduction

Turbulent flows are non-linear systems with a large number of degrees of freedom. In practice, this number is so large that the Navier-Stokes equation is difficult to be solved numerically even by means of the most powerful computers. For that reason, the technique of large-eddy simulation has been used to reduce the computational requirements for simulation of high-Reynolds-number turbulence. In large-eddy simulation, only the large scales (i.e. the low wave-number modes) are directly simulated while the small scales are modeled by introducing an effective viscosity which accounts the effects of small scale motions. Therefore, the technique of large-eddy simulation has to face the problem of modeling the influence of small or sub-grid scales, which is called the *subgrid-modeling* problem.

The attempt to solve the subgrid-modeling problem was initiated by Smagorinsky [52], who introduced the eddy viscosity model (see reference [3] for a more detailed account of the subgrid modeling in large-eddy simulation). Most subgrid modelings in the study of large-eddy simulation are essentially based on phenomenological approaches. It is therefore important to find simple cases in which at least an approximate theory of subgrid modeling might be self-consistently de-

rived. Recently, the renormalization group method has offered a way of dealing with the problem of reducing the number of degrees of freedom necessary to describe turbulent flows at large Reynolds numbers, and of deriving the explicit expression for an effective viscosity from the Navier-Stokes equation.

The renormalization group (hereafter referred to as RG) is, in general, a technique for systematically removing degrees of freedom from the mathematical description of any phenomenon involving many length and time scales. Although the RG method often involves an extensive amount of mathematical techniques from quantum field theory and statistical mechanics, the basic idea is attractively simple. In the study of homogeneous and isotropic turbulence, the RG approach begins with the division of the velocity field into thin shells in Fourier space. Then, the algorithm of RG has two main parts:

(a) *shell elimination*: solve the equation of the motion for retained modes, and the result of shell elimination is added into an effective viscosity;

(b) *re-scaling*: re-scale and repeat this procedure until the effective viscosity does not change. This is known as the fixed point of the calculation.

The first RG study of the Navier-Stokes equation was proposed by Forster, Nelson and Stephen [40, 41] followed after the prescription of Ma and Mazenko [53] for dynamic RG in critical phenomena. However, it has not been proven easy in practice due to the strong coupling that exists between all the modes of turbulent velocity field, and the situation is still not resolved after the first attempt by Forster, Nelson and Stephen. In this chapter, RG methods applied to the study of fluid turbulence are outlined. Attempts to apply RG methods to homogeneous and isotropic turbulence can, in general, be divided into two categories: field-theoretic RG method and recursive-type RG method. The theoretical approach of each type of RG method is discussed. Thus, the motivation for the work given in chapter 3 becomes apparent as the problems found in the following methods are examined.

2.2 Field-theoretic RG method

The earliest attempt to use the RG method for turbulence systems was undertaken by Forster, Nelson and Stephen (referred to as FNS, hereafter) [40, 41]. FNS considered the equation of motion, the Navier-Stokes equation, with a random stirring force (see section 1.2):

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha(\boldsymbol{k},t) = f_\alpha(\boldsymbol{k},t) + \lambda M_{\alpha\beta\gamma}(\boldsymbol{k}) \int d^3j \; u_\beta(\boldsymbol{j},t) u_\gamma(\boldsymbol{k}-\boldsymbol{j},t), \quad (2.1)$$

where a bookkeeping parameter λ has been placed in front of the nonlinear term and $f_\alpha(\boldsymbol{k},t)$ is a centered Gaussian random force with the force correlation function

$$\langle f_\alpha(\boldsymbol{k},t) f_\beta(\boldsymbol{k}',t)\rangle = W(k) D_{\alpha\beta}(\boldsymbol{k}) \delta(\boldsymbol{k}+\boldsymbol{k}'). \quad (2.2)$$

In addition, the specification of the random stirring forces can be completed by the choice of a power law for the force correlation function, thus

$$W(k) = W_0 k^{-y}. \quad (2.3)$$

They also chose the ultraviolet cut-off wave-number to be low enough to excluded cascade effects and established the large-scale and long-time behavior of a randomly stirred fluid. The result of FNS is referred to generically as *stirred hydrodynamics.*

The procedures can be summarized as follows:

1. decompose the velocity field $\boldsymbol{u}$ and the stirring force $\boldsymbol{f}$ into low wave-number and high wave-number parts:

$$u_\alpha(\boldsymbol{k},t) = \begin{cases} u_\alpha^<(\boldsymbol{k},t) & \text{for} \quad 0 < k < b\Lambda \\ u_\alpha^>(\boldsymbol{k},t) & \text{for} \quad b\Lambda < k < \Lambda \end{cases} \quad (2.4)$$

and

$$f_\alpha(\boldsymbol{k},t) = \begin{cases} f_\alpha^<(\boldsymbol{k},t) & \text{for} \quad 0 < k < b\Lambda \\ f_\alpha^>(\boldsymbol{k},t) & \text{for} \quad b\Lambda < k < \Lambda, \end{cases} \quad (2.5)$$

where the scaling parameter b satisfies $0 < b < 1$ and Λ is the ultraviolet cut-off wave-number such that $\Lambda \ll k_d$, the Kolmogorov dissipation wavenumber given by (1.22);

2. eliminate $\boldsymbol{u}^>$ modes which appear in the equation for $\boldsymbol{u}^<$ modes, using a perturbation method;

3. reduce the set of equations by averaging out the effect of the high wave-number modes ($b\Lambda < k < \Lambda$);

4. obtain the new set of equations which are of similar form to the original Navier-Stokes equation. In order to do this step, renormalized coefficients are introduced.

The FNS theory showed that the coupled-mode formalism and the RG method could be employed to obtain new results about properties of the steady states of a forced dissipative system far from equilibrium. In order to avoid the problems encountered due to the strong mode coupling between $\boldsymbol{u}^<$ and $\boldsymbol{u}^>$ modes, FNS restricted their attention to stirring forces which were multivariate normal and excluded the effects of turbulence cascade. Then the *filtered* average was performed (step 3).

However, it has been found that there is a technical problem to average out the effects of the high wave-number modes in terms of the filtered average. If step 1 is carried out, then the corresponding decomposition of the forced Navier-Stokes equation, (2.1), has the form:

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^<(\boldsymbol{k},t) = f_\alpha^<(\boldsymbol{k},t) + \lambda M^<_{\alpha\beta\gamma}(\boldsymbol{k}) \int d^3j \left[u_\beta^<(\boldsymbol{j},t)u_\gamma^<(\boldsymbol{k}-\boldsymbol{j},t) + 2u_\beta^<(\boldsymbol{j},t)u_\gamma^>(\boldsymbol{k}-\boldsymbol{j},t) + u_\beta^>(\boldsymbol{j},t)u_\gamma^>(\boldsymbol{k}-\boldsymbol{j},t)\right] \quad (2.6)$$

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^>(\boldsymbol{k},t) = f_\alpha^>(\boldsymbol{k},t) + \lambda M^>_{\alpha\beta\gamma}(\boldsymbol{k}) \int d^3j \left[u_\beta^<(\boldsymbol{j},t)u_\gamma^<(\boldsymbol{k}-\boldsymbol{j},t) + 2u_\beta^<(\boldsymbol{j},t)u_\gamma^>(\boldsymbol{k}-\boldsymbol{j},t) + u_\beta^>(\boldsymbol{j},t)u_\gamma^>(\boldsymbol{k}-\boldsymbol{j},t)\right]. \quad (2.7)$$

Then, in order to eliminate the effect of the high wave-number modes in equation (2.6), the method requires an assumption that the statistical properties of $\boldsymbol{u}^<$ and $\boldsymbol{f}^<$ should be invariant under the operation of the filtered average, which is in correspondence with the average over the *known* statistic of the force $\boldsymbol{f}^>$. Since the stirring force is chosen to be the multivariate normal mode, $\boldsymbol{f}^<$ can be statistically independent of $\boldsymbol{f}^>$ and be invariant under the filtered averaging process:

$$\langle f_\alpha^<(\boldsymbol{k},t)\rangle_f = f_\alpha^<(\boldsymbol{k},t), \tag{2.8}$$

where $\langle\cdots\rangle_f$ denotes the filtered average. However, the velocity field $\boldsymbol{u}^<$, which is a solution of equation (2.6), does not have the same property, and in general the filtered averaging property of $\boldsymbol{u}^<$ is simply

$$\langle u_\alpha^<(\boldsymbol{k},t)\rangle_f \neq u_\alpha^<(\boldsymbol{k},t). \tag{2.9}$$

In fact, FNS were aware of this point, and stated that the filtered average should really be carried out at constant $\boldsymbol{u}^<$ instead of constant $\boldsymbol{f}^<$. However, it has been pointed out by Eyink [54] in his study of the Ma and Mazenko procedure [53] that the filtered average over $\boldsymbol{f}^>$ that leads to constant $\boldsymbol{u}^<$ should change the statistical distribution of $\boldsymbol{f}^>$ in an *unknown* way, because $\boldsymbol{u}^<$ is *not* statistically *independent* on $\boldsymbol{f}^>$. Again, note that the idea of the filtered average is based on using the known statistic of $\boldsymbol{f}^>$. Nevertheless, Eyink showed that the results of Ma and Mazenko procedure can be regained in his model at the low-orders of perturbation expansion.

Later, the concept of the FNS theory was extended to a theory of turbulence by Yakhot and Orszag [43]. They presented a theory of turbulence with an unusual combination of properties invoking the *correspondence principle*, which states, if the stirring force is chosen to give the Kolmogorov energy spectrum as occurs in real turbulence, then the results from the forced Navier-Stokes equation might also be correct for real turbulence. The results of their work were not only some good numbers for isotropic and homogeneous turbulence, but also the derivation of eddy-viscosity formulas.

However, it seems that their assumption (the correspondence principle) is rarely justifiable. In their work, the exponent of the force correlation function in equation (2.3), y, should be equal to 3, in order to recover the Kolmogorov energy spectrum. This choice, however, leads to a vanishingly small band restriction such that $k_{max}/k_{min} = 1.007$ [3]. Consequently, it does not extend over many orders of wave-number and as such, is not a theory of real turbulence. Furthermore, the ϵ-expansion made in the method, developed by Yakhot and Orszag [43] and reformulated by Yakhot and Smith [46], has also been criticized [3, 55]. They (the authors of references [43] and [46]) assumed that their results obtained when ϵ $(= 4 + y - d)$ is sufficiently small $(\epsilon \ll 1)$ could be relevant when $\epsilon = 4$, in order to retrieve the correspondence principle. However, the justification for allowing this approximation still remains to be explained.

Therefore, it is, as yet, questionable whether the work presented by Yakhot and Orszag is possible to have any relation to real turbulence.

On the other hand, DeDominicis and Martin [42] introduced a RG method in such a way to combine the FNS theory and the Martin-Siggia-Rose functional formalism [35] (see also appendix A). Since then, the method has been extended by others (for the case of a turbulent velocity field: Ronis [44]; Teodorovich [45], and for the case of a passive scalar field: Nayak [47]). However, this method is subject to same restrictions as the work of Yakhot and Orszag mentioned above. In this type of RG method, it is necessary to assume that the numerical results of strongly stirred turbulence and initially cascading turbulence should be equivalent, in order to calculate the turbulent parameters in three-dimensional turbulence (such as the Kolmogorov constant). According to the calculation of DeDominicis and Martin [42], the choice $y = 3$ yields the Kolmogorov energy spectrum, but it must be restricted to a narrow band such that $k_{max}/k_{min} = 1.083$ [3] for arbitrarily setting W_0 in equation (2.3) equal to the dissipation rate of the given system. Thus, their model cannot be extended beyond the ultraviolet cut-off wavenumber.

Although the field-theoretic RG method has produced some good values for representative constants [43, 45] and suggests a hope that it could play the same role in the theory of turbulence as the Ising model does in studies of critical phenomena [53], it seems that some more fundamental justification is required, at least for the case of Navier-Stokes turbulence.

2.3 Recursive-type RG method

2.3.1 Developments

The recursive-type RG method again begins with the decomposition of the velocity field into two parts: low-k (wave-number) and high-k modes. But, there are two substantially different aspects to the starting point:

(a) apply the method to turbulence without introducing multivariate normal model or Gaussian stirring forces: instead, the theory only requires a hypothetical stirring force purely to sustain the turbulent state against viscous dissipation;

(b) perform the finite band elimination from the cut-off wave-number almost same as a dissipation wave-number.

The point (a) therefore indicates that this method has to face the problem which arises due to the non-linear coupling. Note that, in the field-theoretic RG method, this difficulty is evaded by balancing the viscous and stirring terms. Instead, the calculations of the recursive-type RG method are executed in the very high-k domain, where the viscous effects are dominant and hence the non-linear coupling is inherently small.

This approach was first introduced by Rose [48], who considered a perturbative technique to the problem of the convection of a passive scalar field (ϕ) by a frozen velocity field. This gave an additional non-linear term, $\phi \boldsymbol{u}\boldsymbol{u}$, after the finite band

elimination and hence treated the term as part of a new scalar diffusion equation. Later, the work of Rose was extended to the Navier-Stokes equation by McComb and Shanmugasundaram [56] in the course of discussing a new method (see next paragraph), and further studied intensively by Zhou, Vahala and Hossain [50]. Unlike the problem of passive scalar convection, which is essentially a linear problem in terms of $\boldsymbol{u}$, the triple nonlinearity, $\boldsymbol{uuu}$, is eventually a serious obstacle in the velocity field governed by the Navier-Stokes equation. Zhou et al. show a way of maintaining the form invariance of the new equation, which includes the $\boldsymbol{uuu}$ term. They also suggest that it is necessary to supplement the contribution of the triple non-linear term to renormalized viscosity in order to explain a cusp behavior [57] near the cut-off wave-number. However, their procedure still involves uncontrolled approximations to eliminate the effects of the high-k modes. Firstly, the usage of the filtered average (see section 2.2) has been applied to the case of free-decaying turbulence, where the distribution of the velocity field is unknown. Although a modified averaging procedure has been introduced recently [57], it is still questionable whether the filtered average over the velocity field can be carried out, unlike that over the stirring force in the study of FNS. Secondly, in order to preserve the form invariance under RG transformation, their method requires an arbitrary truncation, which has to be repeated in every cycle of iteration.

On the other hand, the attempt of applying the recursive-type RG method to the turbulent velocity field was introduced by McComb [49] via a iterative averaging procedure to be discussed in the following section.

2.3.2 Iterative-averaging RG method

As stated in the previous section, there are two serious obstacles in the way of applying the recursive-type RG method to the velocity-field problem. First, the equation for the high-k modes contains the triple non-linearity, $\boldsymbol{uuu}$, which prevents the form invariance under the RG transformation. The second concern

is the problem of taking an average over the modes that we wish to eliminate.

In fact, the above two difficulties are closely related to each other, because the second may cause the first. As previously mentioned, there is no way to eliminate the high wave-number effects in a simple fashion. Under the circumstance, the method does require an alternative averaging procedure, in order to handle the cross terms involving the product of low-k and high-k modes. This effort was initiated by McComb [49], who considered a short time average over the velocity field. Later, McComb and Shanmugasundaram [56, 58] suggested the *conditional average* in such a way that the low-k modes are held constant, but the average of the high-k modes vanish. Upon using the conditional average, they proposed the iterative-averaging RG method. This method was developed as the *two-field* theory [51]; in particular a more rigorously formulated conditional average was introduced to deal with the coupling of the eliminated and retained modes.

The procedure of the two-field theory presented by McComb and Watt [51] can be summarized as follows:

1. decompose the velocity $\boldsymbol{u}$ into $\boldsymbol{u}^-$ and $\boldsymbol{u}^+$:

$$u_\alpha(\boldsymbol{k},t) = \begin{cases} u_\alpha^-(\boldsymbol{k},t) & \text{for} \quad 0 < k < k_1 \\ u_\alpha^+(\boldsymbol{k},t) & \text{for} \quad k_1 < k < k_0, \end{cases} \tag{2.10}$$

   where k_0 is a maximum wavenumber almost same as a dissipation wavenumber k_d given by (1.22) and k_1 is a cut-off wavenumber defined by

$$k_1 = (1-\eta)k_0 \tag{2.11}$$

   with the bandwidth parameter η satisfying the condition $0 < \eta < 1$.

2. perform the two-field decomposition for high-k modes:

$$u_\alpha^+(\boldsymbol{k},t) = v_\alpha^+(\boldsymbol{k},t) + \Delta_\alpha^+(\boldsymbol{k},t), \tag{2.12}$$

   where $\boldsymbol{v}^+$ is a defined field [it will appear in equation (2.13) later] to be statistically independent of $\boldsymbol{v}^-$, and $\boldsymbol{\Delta}$ is simply the difference between the $\boldsymbol{u}^+$ field and the $\boldsymbol{v}^+$ field;

3. solve $\boldsymbol{u}^-$ equation via the conditional average and the effect of high-k modes leads to a viscosity increment;

4. re-scale the basic variables so that $\boldsymbol{u}^-$ equation looks like the original Navier-Stokes equation;

5. repeat this procedure for $k_2 < k < k_1$, and so on, where $k_2 = (1 - \eta)k_1$.

It should be noted here that McComb and Watt introduced step 2 to evaluate conditionally averaged properties and hence to solve the $\boldsymbol{u}^-$ equation. This point will be dealt with later.

The result of the iterative-averaging RG method, supplemented by the two-field theory, was an equation for the high-k modes that maintains form invariance under the iterative scheme. The calculations showed that the recursion relation for the effective (or renormalized) viscosity reached a fixed point and an acceptable value of the Kolmogorov constant was obtained. However, we have found that there are still some ambiguities which need to be explicitly explained in the procedure of the two-field theory.

Firstly and fundamentally, it raises the question: can the conditional average be actually carried out such that the $\boldsymbol{u}^-$ are held constant, while the average effects of $\boldsymbol{u}^+$ are eliminated? Apparently, the operation of averaging out the $\boldsymbol{u}^+$, while keeping $\boldsymbol{u}^-$ constant, cannot be done rigorously and in itself can only be an approximation. Then, is it a good approximation? In our recent research [59], a direct numerical simulation has been adapted as a tool for probing the basic feasibility of the conditional average. The results, although preliminary, offer crucial support to the hypothesis that a conditional average may be used to reduce the number of degrees of freedom required for the numerical simulation of turbulence.

Secondly, it has been open to criticism, concerning the assumption made by McComb and Watt [51], that the $\boldsymbol{v}^+$ field in equation (2.12) is a first order

Taylor expansion of the $\boldsymbol{u}^+$ field about $k = k_0$, and hence the $\boldsymbol{v}^+$ field has no phase relationship to the $\boldsymbol{u}^-$ field. This was based on the localness assumption of mode-mode interactions in wave-number space within the cascade picture of turbulence. The expression has the form

$$u_\alpha^+(\boldsymbol{k},t) = \underbrace{u_\alpha^+(\boldsymbol{k}_0,t) + (\boldsymbol{k}-\boldsymbol{k}_0)\cdot\boldsymbol{\nabla}_k u_\alpha^+(\boldsymbol{k},t)|_{k=k_0}}_{v_\alpha^+(\boldsymbol{k},t)} + \underbrace{\mathcal{O}\{|\boldsymbol{k}-\boldsymbol{k}_0|^2\}}_{\Delta_\alpha^+(\boldsymbol{k},t)}. \tag{2.13}$$

The magnitude of $\boldsymbol{\Delta}^+$ can be then related to the bandwidth parameter

$$\eta = (k_0 - k_1)/k_0 \tag{2.14}$$

defined in equation (2.11), since the minimum value of k in equation (2.13) is k_1. Consequently, the theory was truncated up-to order of $\boldsymbol{\Delta}^+$, viz

$$\boldsymbol{\Delta}^+ \sim \mathcal{O}\{\eta^2\} \rightarrow \text{ negligible.} \tag{2.15}$$

In general, taking the Taylor expansion of a given function means that the higher orders of Taylor series become negligible by assuming that the slope of two near points does not diverge. However, it might be difficult to say that $\boldsymbol{\Delta}^+$ in equation (2.13) *is* negligible, due to the chaotic trajectory of the Navier-Stokes velocity field. This type of procedure (the use of Taylor series expansion in connection with chaotic trajectories) has been criticized in other areas of mathematical physics. For examples, see references [60] and [61].

Lastly, the two-field theory has suffered from the ambiguity that if the two-field decomposition is made at an earlier stage — in the low-k equation, rather than the high-k equation — then the term which gives rise to the viscosity increment appears to be of a lower order and hence, for consistency, should be neglected. However, on physical grounds, it would be expected that the calculations should not be dependent on performing the two-field decomposition in the low-k equation or the high-k equation.

It is the aim of the work given in this thesis to answer questions arising from the two-field theory [51], and approaches to the problems given in this section are described in the following chapter.

Chapter 3

The Perturbative Implementation of The Iterative-Averaging Renormalization Group Method

3.1 Introduction

As mentioned in the previous chapter, the two-field theory has been developed from a series of papers [51, 62, 63] to find an expression for the effective viscosity and hence a value of the Kolmogorov constant by introducing a conditional average. However, the theory was heuristic, and we have found that there were inadequate explanations in the procedure of that theory. The problems can be found in the two-field theory may be underlined by making questions as follows (see section 2.3.2):

- can we find a *better* expression for the $\boldsymbol{v}^+$ field representing the uncorrelated field between the eliminated band of modes and the remaining modes?
- can we make the two-field decomposition between $\boldsymbol{u}^+$ and $\boldsymbol{v}^+$ in a way that calculations are *independent* of performing the decomposition in the low wave-number equation or high wave-number equation?

In this chapter, we present a systematic re-derivation of the results of the two-field theory of incompressible fluid turbulence [51], by means of a perturbation expansion. In order to achieve this, we introduce a new model of turbulence which represents the $\boldsymbol{v}^+$ field, and is itself based on the two-field decomposition.

The chapter is laid out as follows. In section 2, we give the general algorithm of the RG method, and the basic equations for Navier-Stokes turbulence. Section 3 is concerned with the definition of the conditional average explaining how it may be related to the full ensemble average by introducing a test model field of turbulence. Section 4 is a detailed account of the first band elimination for our RG method. In section 5, we sum up our work by comparing the method of this chapter with that of the two-field theory.

3.2 Formulation and statement of the problem

3.2.1 Basic equations for the iterative averaging method

Consider the turbulent velocity field which is homogeneous, isotropic and stationary, and governed by the forced solenoidal Navier-Stokes equation in wavenumber space:

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha(\boldsymbol{k},t)=\mathcal{F}_\alpha(\boldsymbol{k},t)+M_{\alpha\beta\gamma}(\boldsymbol{k})\int d^3j\; u_\beta(\boldsymbol{j},t)u_\gamma(\boldsymbol{k}-\boldsymbol{j},t), \quad (3.1)$$

where $\mathcal{F}_\alpha(\boldsymbol{k},t)$ is a hypothetical and arbitrary stirring force. However, it should be noted here that it is introduced *not* to act as a basis for perturbation theory, *but* purely to sustain the turbulent state against viscous dissipation.

For the purpose of introducing a statistical treatment, hereafter, we denote the ensemble average of a function $A_\alpha(\boldsymbol{k},t)$ over all possible solutions of the Navier-Stokes equation by $\langle A_\alpha(\boldsymbol{k},t)\rangle$. We also restrict our attention to incompressible fluids subject to turbulent velocity fields with zero mean. As a consequence of these restrictions, we can write the second moment of the turbulent velocity field

as follows:

$$\langle u_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{k}',t')\rangle = Q(k,t-t')D_{\alpha\beta}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}'), \tag{3.2}$$

where $Q(k,t-t')$ is the spectral density and the projector

$$D_{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta k^{-2}, \tag{3.3}$$

which arises due to the incompressibility condition. As stated in chapter 1, the energy spectrum can then be defined as

$$E(k) = 4\pi k^2 Q(k) \tag{3.4}$$

with $Q(k) = Q(k,0)$.

Now consider the velocity field on interval $0 \leq k \leq k_0$, in which the Navier-Stokes equation has the form

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha(\boldsymbol{k},t) = \mathcal{F}_\alpha(\boldsymbol{k},t) + M_{\alpha\beta\gamma}(\boldsymbol{k}) \int_{j<k_0} d^3j \; u_\beta(\boldsymbol{j},t)u_\gamma(\boldsymbol{k}-\boldsymbol{j},t), \tag{3.5}$$

where the maximum cut-off wave-number, k_0, is defined via the dissipation integral [see section 1.2.2]:

$$\varepsilon = \int_0^\infty dk \; 2\nu_0 k^2 E(k) \simeq \int_0^{k_0} dk \; 2\nu_0 k^2 E(k), \tag{3.6}$$

where ε is the dissipation rate and k_0 is of the same order of magnitude as the Kolmogorov dissipation wave-number given by equation (1.22). From the theoretical point of view, the maximum cut-off wave-number k_0 can be chosen arbitrarily, but in practice it is clear that the choice of k_0 would be affected by practical considerations to do with the computational ability for simulations. Since it is the fact that 99.9 % of dissipation takes place at wave-numbers $k < k_d$ [3], the definition of equation (3.6) for k_0 should be enough to describe fluid turbulence.

3.2.2 The renormalization group approach

Now, in order to introduce the RG method, we divide up the velocity field at k_1 as:

$$u_\alpha(\boldsymbol{k},t) = \begin{cases} u_\alpha^-(\boldsymbol{k},t) & \text{for} \quad 0 < k < k_1 \\ u_\alpha^+(\boldsymbol{k},t) & \text{for} \quad k_1 < k < k_0, \end{cases} \tag{3.7}$$

where k_1 is defined by

$$k_1 = (1-\eta)k_0 \tag{3.8}$$

with the bandwidth parameter η satisfying the condition $0 < \eta < 1$.

In order to emphasize the fact that we choose the arbitrary forcing [i.e. $\mathcal{F}$ in equation (3.5)] to be injected at very low wave-number regions, we put the superscript "$\ll$" hereafter, thus $\mathcal{F}^{\ll}$. For example, in our current research of direct numerical simulation [59], we chose the specification of the forcing term as

$$\mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t) = \begin{cases} \varepsilon u_\alpha(\boldsymbol{k},t)/2E_f(t) & \text{if } 0 < k < k_f \\ 0 & \text{otherwise,} \end{cases} \tag{3.9}$$

where the wave-number k_f to be chosen as $k_f/k_0 = 1/80$, ε is the mean dissipation rate, and

$$E_f(t) = \int_0^{k_f} dk\ E(k,t). \tag{3.10}$$

There is no preferred direction in this forcing and the turbulence rapidly reaches a statistically isotropic and steady state.

Now, working with the solenoidal Navier-Stokes equation in k-space given in equation (3.5), we may write the evolution of the low-k velocity field for $0 < k < k_1$ as

$$\begin{aligned} \left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^-(\boldsymbol{k},t) \;=\; & \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t) + M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int^- d^3j\ \Big[u_\beta^-(\boldsymbol{j},t)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t) \\ & + 2u_\beta^-(\boldsymbol{j},t)u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t) + u_\beta^+(\boldsymbol{j},t)u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t)\Big], \end{aligned} \tag{3.11}$$

and the evolution of the high-k velocity field for the first shell, $k_1 < k < k_0$, as

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^+(\boldsymbol{k},t) = M^+_{\alpha\beta\gamma}(\boldsymbol{k}) \int^+ d^3j \left[u_\beta^-(\boldsymbol{j},t)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t) + 2u_\beta^-(\boldsymbol{j},t)u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t) + u_\beta^+(\boldsymbol{j},t)u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t)\right], \tag{3.12}$$

where the superscripts "+" and "−" on $M_{\alpha\beta\gamma}(\boldsymbol{k})$ have the same significance as for $u_\alpha(\boldsymbol{k},t)$, and we have difined

$$\int^- d^3j \equiv \int_{j<k_1} d^3j \quad \text{and} \quad \int^+ d^3j \equiv \int_{k_1<j<k_0} d^3j. \tag{3.13}$$

Note that the forcing term $\mathcal{F}^{\ll}$ appears in equation (3.11) only.

In principle, the RG approach involves two stages:

(i) eliminate the high-k modes, $\boldsymbol{u}^+$, which appear in equation (3.11) for $0 < k < k_1$, by solving for the (mean) effect of the high-k field. This results in an increment to the viscosity, i.e. $\nu_0 \to \nu_1 = \nu_0 + \delta\nu_0$;

(ii) re-scale the basic variables, so that the Navier-Stokes equation for $0 < k < k_1$ looks like the original Navier-Stokes equation for $0 < k < k_0$.

Although this procedure is appealingly simple, it has not proved easy to put into practice in the turbulence problem. A typical approach is to eliminate all the high-k effects in equation (3.11), by substituting the solution of equation (3.12), directly into the $\boldsymbol{u}^+$ modes in the $\boldsymbol{u}^-$ equation. However, problems are then encountered because of the mode coupling between $\boldsymbol{u}^-$ and $\boldsymbol{u}^+$. Even if one succeeds in carrying out the first part, the further problem of averaging out the high-k modes arises immediately, because $\boldsymbol{u}^-$ and $\boldsymbol{u}^+$ are not statistically independent. This problem was avoided by Foster, Nelson and Stephen [40] in their pioneering study of stirred fluid motion, as they restricted their attention to stirring forces which were multivariate normal and excluded the effects of

the turbulence cascade. In the next section, we shall consider the more general situation by using a formal conditional average [63].

The first problem, that of solving for $\boldsymbol{u}^+$, is normally tackled by setting up a perturbation expansion in the form

$$\boldsymbol{u}^+ = \boldsymbol{u}^{+(0)} + \lambda \boldsymbol{u}^{+(1)} + \lambda^2 \boldsymbol{u}^{+(2)} + \cdots, \tag{3.14}$$

where λ is a book-keeping parameter which is put equal to unity at the end of the calculation. As is well known, the success or otherwise of such a procedure depends on the choice of a tractable form of model for the zero-order field (for more general discussions, see reference [3]). In the following section, we shall introduce a model field which we shall call $\boldsymbol{v}^+$, and which will be used to represent $\boldsymbol{u}^{+(0)}$ in the above expansion.

3.3 The conditional average and the model field

3.3.1 The definition of the conditional average

The velocity field $\boldsymbol{u}$ is a representative member of the ensemble, which consists of a large set of realizations carried out under similar initial conditions. Formally, we may define our ensemble as the set $\mathcal{X}$, where

$$\mathcal{X} = \{X_\alpha^{(n)}(\boldsymbol{k}, t) | \alpha = 1, 2, 3; 0 < k < k_{max}; n = 1, \ldots, N\}, \tag{3.15}$$

where each realization $X_\alpha^{(n)}(\boldsymbol{k}, t)$ is a solution of Navier-Stokes equation and the superscript n denotes the specific initial condition for the given $X_\alpha^{(n)}(\boldsymbol{k}, t)$. Note that each member of the set $\mathcal{X}$ will be same, if we specify the same initial conditions.

We can now define the ensemble average in a way:

$$\langle F\{u_\alpha(\boldsymbol{k}, t)\}\rangle = \frac{1}{N} \sum_{n=1}^{N} F\{X_\alpha^{(n)}(\boldsymbol{k}, t)\}, \tag{3.16}$$

where $\langle \cdots \rangle$ represents the ensemble average and $F\{u_\alpha(\boldsymbol{k},t)\}$ is an arbitrary functional of $u_\alpha(\boldsymbol{k},t)$.

In order to carry out RG calculations, it is necessary to introduce a conditional average which averages out the effect of high-k modes while keeping the $\boldsymbol{u}^-$ (approximately) constant. In principle, we may do this by extracting a sub-ensemble set $\mathcal{W} \subset \mathcal{X}$, which is a subset of the whole ensemble with M ($M \leq N$) members:

$$\mathcal{W} = \{W_\alpha^{(m)}(\boldsymbol{k},t) | \alpha = 1,2,3; 0 < k < k_{max}; m = 1,\ldots,M\}, \tag{3.17}$$

chosen such that

$$\left|\theta(k_c - k)W_\alpha^{(i)}(\boldsymbol{k},t) - \theta(k_c - k)W_\alpha^{(j)}(\boldsymbol{k},t)\right| \leq \xi \tag{3.18}$$

for $i \neq j$, for an arbitrarily small value of ξ, and for each realization $W^{(i)} \in \mathcal{W}$ and $W^{(j)} \in \mathcal{W}$ (see also figure 3.1). Here, the step function $\theta(k_c - k)$ is

$$\theta(k_c - k) = \begin{cases} 1 & \text{if} \quad k \leq k_c \\ 0 & \text{if} \quad k \geq k_c \end{cases} \tag{3.19}$$

Now, we can also define the conditional average over the sub-ensemble set $\mathcal{W}$ in the obvious way:

$$\langle F\{u_\alpha(\boldsymbol{k},t)\}\rangle_c = \frac{1}{M}\sum_{m=1}^{M} F\{W_\alpha^{(m)}(\boldsymbol{k},t)\}, \tag{3.20}$$

where $\langle \cdots \rangle_c$ stands for the conditional average.

For our present purpose, we take the velocity field $u_\alpha(\boldsymbol{k},t)$ to be a prescribed function of $\boldsymbol{k}$ and t, and one of members of the sub-ensemble set:

$$u_\alpha(\boldsymbol{k},t) = W_\alpha^{(\Re)}(\boldsymbol{k},t) \text{ and } W_\alpha^{(\Re)}(\boldsymbol{k},t) \in \mathcal{W}. \tag{3.21}$$

Then the properties of the conditional averages involving the low-k modes may be listed as follows:

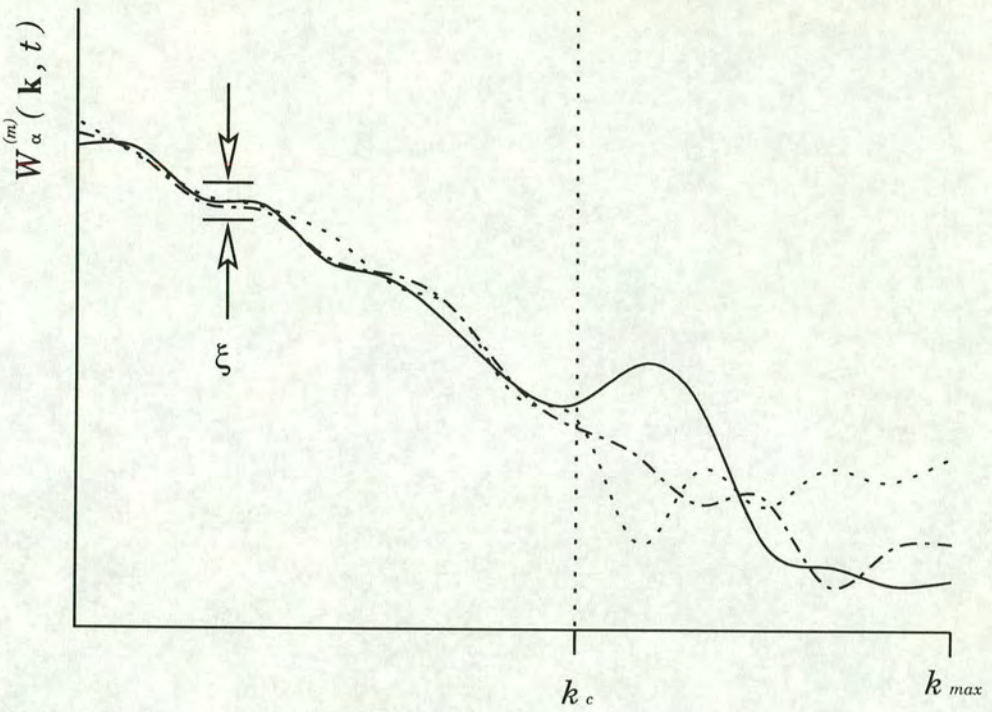

Figure 3.1: A schematic illustration of the way realizations are chosen for the sub-ensemble set $\mathcal{W}$, in order to satisfy the restriction given by equation (3.18).

$$\begin{aligned}\langle u_\alpha^-(\boldsymbol{k},t)\rangle_c &= \frac{1}{M}\sum_{m=1}^{M}\theta(k_c-k)W_\alpha^{(m)}(\boldsymbol{k},t)\\ &= u_\alpha^-(\boldsymbol{k},t)+\mathcal{O}\{\xi\}; \qquad (3.22)\\ \langle u_\alpha^-(\boldsymbol{k},t)u_\beta^-(\boldsymbol{k}',t')\rangle_c &= \frac{1}{M}\sum_{m=1}^{M}\theta(k_c-k)W_\alpha^{(m)}(\boldsymbol{k},t)\theta(k_c-k')W_\alpha^{(m)}(\boldsymbol{k}',t')\\ &= u_\alpha^-(\boldsymbol{k},t)u_\beta^-(\boldsymbol{k}',t')+\mathcal{O}\{\xi\}. \qquad (3.23)\end{aligned}$$

Therefore, we now have an operation in which the low-k modes of Navier-Stokes velocity field can be held approximately constant. However, the problem of knowing how to develop an operation of taking the conditional average involving $\boldsymbol{u}^+$ is still with us, and this will be the subject of the following sections.

3.3.2 The model field

Our objective is now to find a model field for which the conditional average of the high-k modes can be evaluated exactly. In order to do this, let us introduce a particular field, $\boldsymbol{v}$, which is obtained by choosing one member of the ensemble (not, of course, $\boldsymbol{u}$) to provide its low-k modes and an unconnected realization to provide its high-k modes [the definition of $\boldsymbol{v}^+$ will appear in (3.27) later].

Each member of the set $\mathcal{W}$ has the same statistical properties but in general we choose initial conditions arbitrarily, and hence there is no fixed phase phase relationship between any two members of the set (see also Figure 3.1). For the particular field $\boldsymbol{v}$, furthermore, there is no phase relationship between $\boldsymbol{u}^-$ and $\boldsymbol{v}^-$, nor between $\boldsymbol{u}^+$ and $\boldsymbol{v}^+$, nor (most importantly) between $\boldsymbol{v}^-$ and $\boldsymbol{v}^+$. This is the case that we shall take $\boldsymbol{v}$ as our zero-order model.

We now generate our model field as follow. First, we decompose the $\boldsymbol{v}$ field as

$$v_\alpha(\boldsymbol{k},t) = \begin{cases} v_\alpha^-(\boldsymbol{k},t) & \text{for} \quad 0 < k < k_1 \\ v_\alpha^+(\boldsymbol{k},t) & \text{for} \quad k_1 < k < k_0. \end{cases} \tag{3.24}$$

The characteristics of the $\boldsymbol{v}$ field are then as follows:

- The low-k mode of $\boldsymbol{v}$ is same member of the sub-ensemble set:

$$v_\alpha^-(\boldsymbol{k},t) = \theta(k_1 - k) W_\alpha^{(i)}(\boldsymbol{k},t) \text{ and } W_\alpha^{(i)}(\boldsymbol{k},t) \in \mathcal{W}. \tag{3.25}$$

  Note that $\boldsymbol{u}^-$ is also

$$u_\alpha^-(\boldsymbol{k},t) = \theta(k_1 - k) W_\alpha^{(\Re)}(\boldsymbol{k},t) \text{ and } W_\alpha^{(\Re)}(\boldsymbol{k},t) \in \mathcal{W}, \tag{3.26}$$

  but $i \neq \Re$.

- For high-k mode of $\boldsymbol{v}$, we shall introduce a new dynamical equation:

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] v_\alpha^+(\boldsymbol{k},t)$$
$$= M_{\alpha\beta\gamma}^+(\boldsymbol{k}) \int^+ d^3j \left[v_\beta^-(\boldsymbol{j},t) v_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t) + v_\beta^+(\boldsymbol{j},t) v_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t)\right]. \tag{3.27}$$

The ansatz for equation (3.27) is based on the idea that the turbulent energy transfer between various scales in wave-number takes the form of a cascade and hence is *local*. In other words, the coupling terms between the high-k and low-k modes of the Navier-Stokes velocity field do not play the dominant role in energy transfer between scales. In fact, we are imposing here that if the particular mode-mode coupling, between the high-k and low-k modes of the velocity field, induced by the Navier-Stokes equation can be neglected in our particular model field, then the conditional average of the high-k modes of $\boldsymbol{v}$ would reduce to the full ensemble average; hence the correction term, the phase difference between $\boldsymbol{u}^+$ and $\boldsymbol{v}^+$, represents the effect of that coupling (see also section 3.5 later).

Our next step is to introduce an operation of taking the conditional average of the $\boldsymbol{v}$ field. For the model field, even if we choose a deterministic sub-ensemble set for $0 < k < k_1$, in which we can hold $\boldsymbol{v}^-$ constant, the statistically independent $\boldsymbol{v}^+$ field is unconstrained and the conditional average is exactly the same as a filtered ensemble average (see figure 3.2). Therefore, the properties of the $\boldsymbol{v}$ field under the conditional average are:

- The low-k modes of $\boldsymbol{v}$ are held constant, i.e.

$$\langle \boldsymbol{v}^- \rangle_c = \boldsymbol{v}^-, \tag{3.28}$$

  where $\langle \cdots \rangle_c$ denotes the conditional average over the sub-ensembles;

- The conditional average of the high-k modes of $\boldsymbol{v}$ is same as the ensemble average because of the lack of the coupling between $\boldsymbol{v}^-$ and $\boldsymbol{v}^+$, that is

$$\langle \boldsymbol{v}^+ \rangle_c = \langle \boldsymbol{v}^+ \rangle = 0. \tag{3.29}$$

3.3.3 Evaluation of the conditional average in terms of the representative model field

The leading concern is now to express the conditional average of the $\boldsymbol{u}$ field in terms of the previously given properties of the model field. We begin by noting

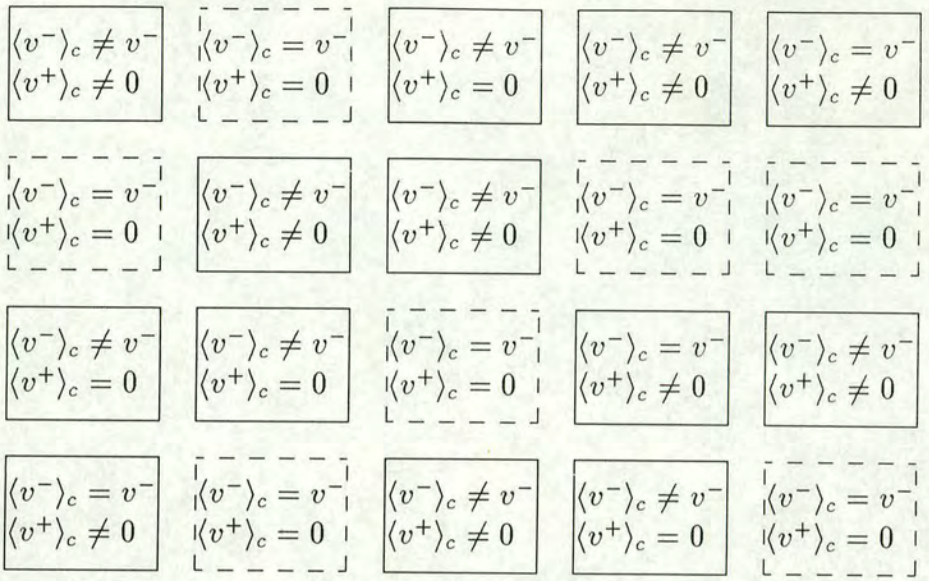

Figure 3.2: A schematic illustration of the way of choosing the sub-ensemble (dashed boxes) from the micro-canonical ensemble (all boxes) under the conditional average.

that, for Navier-Stokes turbulence, if we hold $\boldsymbol{u}^-$ constant then $\boldsymbol{u}^+$ must also be constant due to the deterministic nature of the Navier-Stokes equation. Thus it is not enough to simply set the low-k parts of the Navier-Stokes and model fields equal to each other. Accordingly, as in the Two-field theory [51], we relate the Navier-Stokes field, $\boldsymbol{u}$, and the model field, $\boldsymbol{v}$, by writing

$$\boldsymbol{u}^- = \boldsymbol{v}^- + \boldsymbol{\Phi}^-, \tag{3.30}$$

$$\boldsymbol{u}^+ = \boldsymbol{v}^+ + \boldsymbol{\Delta}^+. \tag{3.31}$$

Note that although $\boldsymbol{\Phi}^-$ and $\boldsymbol{\Delta}^+$ are similar from the mathematical point of view, they have different physical interpretations, thus:

$\boldsymbol{\Phi}^-$ is the uncertainty between the $\boldsymbol{u}^-$ realization and the $\boldsymbol{v}^-$ resulting from the arbitrary choice of a particular model field $\boldsymbol{v}$;

$\boldsymbol{\Delta}^+$ is the growing difference in the high-k band generated by the uncertainty $\boldsymbol{\Phi}^-$, and results from the chaotic nature of the Navier-Stokes equation.

Let us now express conditional averages over the $\boldsymbol{u}$ field in terms of the operations defined in equations (3.28) and (3.29). First, for the low-k modes, we get

$$\langle \boldsymbol{u}^- \rangle_c = \boldsymbol{v}^- + \langle \boldsymbol{\Phi}^- \rangle_c, \tag{3.32}$$

where $\boldsymbol{\Phi}^-$ is subject to the weak constraint imposed by the requirement that $\boldsymbol{u}^-$ and $\boldsymbol{v}^-$ must both be solutions of the Navier-Stokes equation. By properly choosing the arbitrary $\boldsymbol{v}$ field, $\boldsymbol{\Phi}^-$ can be chosen to satisfy $\langle \boldsymbol{\Phi}^- \rangle_c = 0$ so that equation (3.32) becomes

$$\langle \boldsymbol{u}^- \rangle_c = \boldsymbol{v}^-. \tag{3.33}$$

However, operating the conditional average on the high-k modes of the Navier-Stokes equation is a different story. From equations (3.29) and (3.31), we have

$$\langle \boldsymbol{u}^+ \rangle_c = \langle \boldsymbol{\Delta}^+ \rangle_c, \tag{3.34}$$

where the properties of $\langle \boldsymbol{\Delta}^+ \rangle_c$ are unknown at this stage.

Accepting for the moment that we still have the problem of how to handle $\boldsymbol{\Delta}^+$, such an average over the sub-ensemble which satisfies (3.28) and (3.29) will have properties as follows:

$$\langle \boldsymbol{u}^- \boldsymbol{u}^- \rangle_c = \boldsymbol{v}^- \boldsymbol{v}^- + \langle \boldsymbol{\Phi}^- \boldsymbol{\Phi}^- \rangle_c \cong \boldsymbol{v}^- \boldsymbol{v}^-; \tag{3.35}$$

$$\langle \boldsymbol{u}^- \boldsymbol{u}^+ \rangle_c = \boldsymbol{v}^- \langle \boldsymbol{\Delta}^+ \rangle_c + \langle \boldsymbol{\Phi}^- \boldsymbol{\Delta}^+ \rangle_c; \tag{3.36}$$

$$\langle \boldsymbol{u}^+ \boldsymbol{u}^+ \rangle_c = \langle \boldsymbol{v}^+ \boldsymbol{v}^+ \rangle + 2\langle \boldsymbol{v}^+ \boldsymbol{\Delta}^+ \rangle_c + \langle \boldsymbol{\Delta}^+ \boldsymbol{\Delta}^+ \rangle_c. \tag{3.37}$$

Note that we neglect $\langle \boldsymbol{\Phi}^- \boldsymbol{\Phi}^- \rangle_c$ in (3.35) as a small quantity. It and the higher-order terms will be treated as errors in the rest of this paper. What we are, in effect, saying is that the properties of the Navier-Stokes equation are such that only a very small uncertainty in the deterministic specification of the low-k modes is sufficient to generate chaos in the high-k modes.

3.4 Procedure: first shell elimination

3.4.1 High wave-number equation: perturbation expansion

Now, we shall give a formulation to handle $\boldsymbol{\Delta}^+$ in detail, by means of perturbation theory. In order to do this, we rewrite equation (3.31) as

$$\boldsymbol{u}^+ = \boldsymbol{v}^+ + \lambda \boldsymbol{\Delta}^+, \tag{3.38}$$

where λ is the book-keeping parameter introduced in equation (3.38) and will be set equal to unity at the end of the calculation. An explicit form of (3.38) can be obtained by subtracting equation (3.27) from equation (3.12), thus:

$$\begin{aligned} u_\alpha^+(\boldsymbol{k},t) &= v_\alpha^+(\boldsymbol{k},t) + \lambda M^+_{\alpha\beta\gamma}(\boldsymbol{k}) \int^+ d^3j \int_{-\infty}^{t} dt'\, G^{(0)}(k;t,t') \\ &\quad \times \Big[u_\beta^-(\boldsymbol{j},t') u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t') + 2u_\beta^-(\boldsymbol{j},t') u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad + u_\beta^+(\boldsymbol{j},t') u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t') - v_\beta^-(\boldsymbol{j},t') v_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad - v_\beta^+(\boldsymbol{j},t') v_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t') \Big], \end{aligned} \tag{3.39}$$

where we have inserted a factor λ (=1, of course!) in order to make it consistent with equation (3.38) and the Green's function $G^{(0)}(k;t,t')$ is given by

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2 \right] G^{(0)}(k;t,t') = \delta(t-t') \tag{3.40}$$

with the solution

$$G^{(0)}(k;t,t') = \exp\left[-\nu_0 k^2 (t-t') \right]. \tag{3.41}$$

In principle, we can use the zeroth-order solution of equation (3.39) by taking $\boldsymbol{u}^{+(0)} = \boldsymbol{v}^+$ to construct the perturbation expansion of $\boldsymbol{u}^+$ in power of λ. We can therefore solve the equation for high-k modes, (3.39), with the results as follows:

$$u_\alpha^{+(0)}(\boldsymbol{k},t) = v_\alpha^+(\boldsymbol{k},t); \tag{3.42}$$

$$u_\alpha^{+(1)}(\boldsymbol{k},t) = M^+_{\alpha\beta\gamma}(\boldsymbol{k}) \int^+ d^3j \int_{-\infty}^{t} dt'\, G^{(0)}(k;t,t') \Big[u_\beta^-(\boldsymbol{j},t') u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t') -$$

$$-v_\beta^-(\boldsymbol{j},t')v_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t')+2u_\beta^-(\boldsymbol{j},t')u_\gamma^{+(0)}(\boldsymbol{k}-\boldsymbol{j},t')\Big]; \tag{3.43}$$

$$u_\alpha^{+(2)}(\boldsymbol{k},t) = M^+_{\alpha\beta\gamma}(\boldsymbol{k})\int^+ d^3j \int_{-\infty}^{t} dt'\, G^{(0)}(k;t,t')\Big[2u_\beta^-(\boldsymbol{j},t')u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t') +2u_\beta^{+(0)}(\boldsymbol{j},t')u_\gamma^{+(1)}(\boldsymbol{k}-\mathbf{j},t')\Big], \tag{3.44}$$

and so on.

3.4.2 Low wave-number equation

By substituting the perturbation expansion for $\boldsymbol{u}^+$ into equation (3.11) for the low-k modes, we have the equation up-to order of λ as

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu_0k^2\right]u_\alpha^-(\boldsymbol{k},t) &= \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t)+M^-_{\alpha\beta\gamma}(\boldsymbol{k})\int^- d^3j\,\Big[u_\beta^-(\boldsymbol{j},t)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t)\\ &\quad +2u_\beta^-(\boldsymbol{j},t)u_\gamma^{+(0)}(\boldsymbol{k}-\boldsymbol{j},t)+u_\beta^{+(0)}(\boldsymbol{j},t)u_\gamma^{+(0)}(\boldsymbol{k}-\boldsymbol{j},t)\Big]\\ &\quad +\lambda M^-_{\alpha\beta\gamma}(\boldsymbol{k})\int^- d^3j\,\Big[2u_\beta^-(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\\ &\quad +2u_\beta^{+(0)}(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\Big]+\mathcal{O}\{\lambda^2\}.\end{aligned} \tag{3.45}$$

At this stage, we replace each $\boldsymbol{u}^-$ with $\boldsymbol{v}^-+\boldsymbol{\Phi}^-$ in equation (3.45), thus:

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\nu_0k^2\right]\Big\{v_\alpha^-(\boldsymbol{k},t)+\Phi_\alpha^-(\boldsymbol{k},t)\Big\}\\ &= \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t)+M^-_{\alpha\beta\gamma}(\boldsymbol{k})\int^- d^3j\,\Big[v_\beta^-(\boldsymbol{j},t)v_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t)+2v_\beta^-(\boldsymbol{j},t)\Phi_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t)\\ &\quad +\Phi_\beta^-(\boldsymbol{j},t)\Phi_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t)+2v_\beta^-(\boldsymbol{j},t)u_\gamma^{+(0)}(\boldsymbol{k}-\boldsymbol{j},t)\\ &\quad +2\Phi_\beta^-(\boldsymbol{j},t)u_\gamma^{+(0)}(\boldsymbol{k}-\boldsymbol{j},t)+u_\beta^{+(0)}(\boldsymbol{j},t)u_\gamma^{+(0)}(\boldsymbol{k}-\boldsymbol{j},t)\Big]\\ &\quad +\lambda M^-_{\alpha\beta\gamma}(\boldsymbol{k})\int^- d^3j\,\Big[2v_\beta^-(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)+2\Phi_\beta^-(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\\ &\quad +2u_\beta^{+(0)}(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\Big],\end{aligned} \tag{3.46}$$

and taking the conditional average of each term of equation (3.46) gives

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] v_\alpha^-(\boldsymbol{k},t)$$

$$= \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t) + M^-_{\alpha\beta\gamma}(\boldsymbol{k}) \int^- d^3j \, \Big[v_\beta^-(\boldsymbol{j},t) v_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t)$$

$$+\langle \Phi_\beta^-(\boldsymbol{j},t)\Phi_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c\Big]$$

$$+\lambda M^-_{\alpha\beta\gamma}(\boldsymbol{k}) \int^- d^3j \, \Big[2v_\beta^-(\boldsymbol{j},t)\langle u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c$$

$$+2\langle \Phi_\beta^-(\boldsymbol{j},t) u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c + 2\langle u_\beta^{+(0)}(\boldsymbol{j},t) u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c\Big]. \quad (3.47)$$

Note that $\mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t)$ should be invariant under the conditional average defined in section 3.3, i.e.

$$\langle \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t)\rangle_c = \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t). \quad (3.48)$$

Now, substituting each $\boldsymbol{u}^{+(1)}$ in equation (3.47) with the RHS of equation (3.43) gives

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] v_\alpha^-(\boldsymbol{k},t)$$

$$= \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t) + M^-_{\alpha\beta\gamma}(\boldsymbol{k}) \int^- d^3j \, \Big[v_\beta^-(\boldsymbol{j},t) v_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t)$$

$$+\langle \Phi_\beta^-(\boldsymbol{j},t)\Phi_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c\Big]$$

$$+2\lambda M^-_{\alpha\beta\gamma}(\boldsymbol{k}) \int^- d^3j \; M^+_{\gamma\delta\sigma}(\boldsymbol{k}-\boldsymbol{j}) \int^+ d^3p \int_{-\infty}^{t} dt \; G^{(0)}(|\boldsymbol{k}-\boldsymbol{j}|;t,t')$$

$$\times\Big[v_\beta^-(\boldsymbol{j},t)\langle \Phi_\delta^-(\boldsymbol{p},t')\Phi_\sigma^-(\boldsymbol{p}-\boldsymbol{k}+\boldsymbol{j},t')\rangle_c$$

$$+2v_\delta^-(\boldsymbol{p},t')\langle \Phi_\beta^-(\boldsymbol{j},t)\Phi_\sigma^-(\boldsymbol{p}-\boldsymbol{k}+\boldsymbol{j},t')\rangle_c$$

$$+\langle \Phi_\beta^-(\boldsymbol{j},t)\Phi_\delta^-(\boldsymbol{p},t')\Phi_\sigma^-(\boldsymbol{p}-\boldsymbol{k}+\boldsymbol{j},t')\rangle_c\Big]$$

$$+2\lambda M^-_{\alpha\beta\gamma}(\boldsymbol{k}) \int^- d^3j \; \langle u_\beta^{+(0)}(\boldsymbol{j},t) u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c. \quad (3.49)$$

It should be noted here that the calculation for the last term of the RHS of equation (3.49) has not been carried out here. This will be treated in the following section.

Then, we now have the key equation for the low-k modes by rewriting equation (3.49) as

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] v_\alpha^-(\boldsymbol{k},t) = \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t)+M_{\alpha\beta\gamma}^-(\boldsymbol{k})\int^- d^3j\; v_\beta^-(\boldsymbol{j},t)v_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t)$$
$$+2\lambda M_{\alpha\beta\gamma}^-(\boldsymbol{k})\int^- d^3j\; \langle u_\beta^{+(0)}(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c$$
$$+\mathcal{E}\{\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c\}+\mathcal{O}\{\lambda^2\}, \qquad (3.50)$$

where $\mathcal{E}\{\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c\}$ indicates the error terms which contain $\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c$ and higher orders. Note that, as mentioned in section 3.2.2, $\mathcal{E}\{\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c\}$ will be neglected as a small quantity at the end of calculations.

3.4.3 Boundary-layer-type approximations

At this stage, all the high-k effects in equation (3.50) for the low-k modes, are represented in the term:

$$2\lambda M_{\alpha\beta\gamma}^-(\boldsymbol{k})\int^- d^3j\; \langle u_\beta^{+(0)}(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c. \qquad (3.51)$$

In principle, this term can be evaluated by direct substitution of $\boldsymbol{u}^{+(1)}$. However, it has been found that this straightforward approach does not lead to controlled approximations: we shall return to this point later (see section 4.3). An alternative procedure, which allows us to carry the work further, is to form an equation for $\boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(1)}$. In order to do this, we take the following steps:

(i) rewrite equation (3.27) for $[\partial_t+\nu_0 j^2]u_\beta^{+(0)}(\boldsymbol{j},t)$ on the LHS and multiply it through by $u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)$: note that $\boldsymbol{u}^{+(0)}\equiv\boldsymbol{v}^+$;

(ii) rewrite equation (3.43) for $[\partial_t+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2]u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)$ and multiply it by $u_\beta^{+(0)}(\boldsymbol{j},t)$;

(iii) add the two equations produced by steps (i) and (ii).

This procedure yields

$$\begin{aligned}
&\left[\frac{\partial}{\partial t}+\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2\right] u_\beta^{+(0)}(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t) \\
&= M^+_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})\int^+ d^3p\,\Big[2u_\beta^{+(0)}(\boldsymbol{j},t)u_\rho^-(\boldsymbol{p},t)u_\sigma^-(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t) \\
&\quad -u_\beta^{+(0)}(\boldsymbol{j},t)v_\rho^-(\boldsymbol{p},t)v_\sigma^-(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t) \\
&\quad +2u_\beta^{+(0)}(\boldsymbol{j},t)u_\rho^-(\boldsymbol{p},t)u_\sigma^{+(0)}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t)\Big] \\
&\quad +M^+_{\beta\delta\epsilon}(\boldsymbol{j})\int^+ d^3q\,\Big[v_\delta^-(\boldsymbol{q},t)v_\epsilon^-(\boldsymbol{j}-\boldsymbol{q},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t) \\
&\quad +u_\delta^{+(0)}(\boldsymbol{q},t)u_\epsilon^{+(0)}(\boldsymbol{j}-\boldsymbol{q},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\Big]. \qquad (3.52)
\end{aligned}$$

Now, replacing each $\boldsymbol{u}^-$ with $\boldsymbol{v}^- + \boldsymbol{\Phi}^-$ in equation (3.52):

$$\begin{aligned}
&\left[\frac{\partial}{\partial t}+\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2\right] u_\beta^{+(0)}(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t) \\
&= M^+_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})\int^+ d^3p\,\Big[2u_\beta^{+(0)}(\boldsymbol{j},t)v_\rho^-(\boldsymbol{p},t)v_\sigma^-(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t) \\
&\quad +4u_\beta^{+(0)}(\boldsymbol{j},t)v_\rho^-(\boldsymbol{p},t)\Phi_\sigma^-(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t) \\
&\quad +2u_\beta^{+(0)}(\boldsymbol{j},t)\Phi_\rho^-(\boldsymbol{p},t)\Phi_\sigma^-(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t) \\
&\quad -u_\beta^{+(0)}(\boldsymbol{j},t)v_\rho^-(\boldsymbol{p},t)v_\sigma^-(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t) \\
&\quad +2u_\beta^{+(0)}(\boldsymbol{j},t)v_\rho^-(\boldsymbol{p},t)u_\sigma^{+(0)}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t) \\
&\quad +2u_\beta^{+(0)}(\boldsymbol{j},t)\Phi_\rho^-(\boldsymbol{p},t)u_\sigma^{+(0)}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t)\Big] \\
&\quad +M^+_{\beta\delta\epsilon}(\boldsymbol{j})\int^+ d^3q\,\Big[v_\delta^-(\boldsymbol{q},t)v_\epsilon^-(\boldsymbol{j}-\boldsymbol{q},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t) \\
&\quad +u_\delta^{+(0)}(\boldsymbol{q},t)u_\epsilon^{+(0)}(\boldsymbol{j}-\boldsymbol{q},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\Big], \qquad (3.53)
\end{aligned}$$

and taking the conditional average of each term in equation (3.53) gives

$$\begin{aligned}
&\left[\frac{\partial}{\partial t}+\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2\right] \langle u_\beta^{+(0)}(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c \\
&= 2M^+_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})\int^+ d^3p\,\langle u_\beta^{+(0)}(\boldsymbol{j},t)u_\sigma^{+(0)}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t)\rangle_c v_\rho^-(\boldsymbol{p},t)+
\end{aligned}$$

$$+M^{+}_{\beta\delta\epsilon}(\boldsymbol{j})\int^{+} d^3q\ \langle u^{+(0)}_{\delta}(\boldsymbol{q},t)u^{+(0)}_{\epsilon}(\boldsymbol{j}-\boldsymbol{q},t)u^{+(1)}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c$$

$$+\mathcal{E}\{\langle \boldsymbol{\Phi}^{-}\boldsymbol{\Phi}^{-}\rangle_c\}. \tag{3.54}$$

Hereafter, for compactness, the term $\mathcal{E}\{\langle \boldsymbol{\Phi}^{-}\boldsymbol{\Phi}^{-}\rangle_c\}$ is discarded.

In order to investigate the work further, we now have to make two approximations that seem to be justified on physical grounds. If we briefly revert to real space, two aspects of such a decomposition may be seen as helpful in carrying out a calculation which eliminates the high-k modes. Let us take $u^{-}_{\alpha}(\boldsymbol{x},t)$ and $u^{+}_{\alpha}(\boldsymbol{x},t)$ to be the Fourier transforms of $u^{-}_{\alpha}(\boldsymbol{k},t)$ and $u^{+}_{\alpha}(\boldsymbol{k},t)$, respectively. Then, from our experimental knowledge that the turbulence spectrum is a rapidly decreasing function of k, with most of its energy contained in low-k modes where the dissipation rate is negligible, we may assume that

$$\left\langle [u^{+}_{\alpha}(\boldsymbol{x},t)]^2 \right\rangle \ll \left\langle [u^{-}_{\alpha}(\boldsymbol{x},t)]^2 \right\rangle. \tag{3.55}$$

So far as the high-k modes are concerned, this allows us to deal with the notorious closure problem. Second, we may expect that $u^{+}_{\alpha}(\boldsymbol{x},t)$ will evolve rapidly on time-scales associated with the $u^{-}_{\alpha}(\boldsymbol{x},t)$. As we shall see later on, this will be helpful in permitting us to introduce a Markovian type of approximation. McComb and Watt [51] introduced the above two aspects as *boundary-layer* type approximations:

1. The velocity components in the high-k band are much smaller than the velocity components in the low-k band;

2. The velocity modes in the high-k band will evolve much more rapidly than the velocity modes in the low-k band.

Now, the first approximation allows us to drop the triple moment of the form $\langle \boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(1)}\rangle_c$ in comparison to $\boldsymbol{v}^{-}\langle \boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(0)}\rangle_c$ and, accordingly, write equation (3.54) as

$$\left[\frac{\partial}{\partial t}+\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2\right]\langle u^{+(0)}_{\beta}(\boldsymbol{j},t)u^{+(1)}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c =$$

$$= 2M^{+}_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})\int^{+} d^3p\ \langle u^{+(0)}_{\beta}(\boldsymbol{j},t)u^{+(0)}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t)\rangle_c v^{-}_{\rho}(\boldsymbol{p},t) \quad (3.56)$$

Invoking that $\boldsymbol{u}^{+(0)} = \boldsymbol{v}^{+}$ and the $\boldsymbol{v}^{+}$ field is statistically homogeneous, isotropic and stationary, we can also write

$$\langle u^{+(0)}_{\beta}(\boldsymbol{j},t)u^{+(0)}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t)\rangle = \langle v^{+}_{\beta}(\boldsymbol{j},t)v^{+}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t)\rangle_c \quad (3.57)$$

$$= D_{\beta\sigma}(\boldsymbol{j})Q^{+}_{v}(j)\delta(\boldsymbol{k}-\boldsymbol{p}). \quad (3.58)$$

Substituting from equation (3.58) into equation (3.56) give us

$$\left[\frac{\partial}{\partial t} + \nu_0 j^2 + \nu_0|\boldsymbol{k}-\boldsymbol{j}|^2\right]\langle u^{+(0)}_{\beta}(\boldsymbol{j},t)u^{+(1)}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c$$
$$= 2M^{+}_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})D_{\beta\sigma}(\boldsymbol{j})Q^{+}_{v}(j)v^{-}_{\rho}(\boldsymbol{k},t) \quad (3.59)$$

or

$$\langle u^{+(0)}_{\beta}(\boldsymbol{j},t)u^{+(1)}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c = 2M^{+}_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})D_{\beta\sigma}(\boldsymbol{j})Q^{+}_{v}(j)\int_0^{\infty} d\tau$$
$$\times \exp\Big[-(\nu_0 j^2 + \nu_0|\boldsymbol{k}-\boldsymbol{j}|^2)\tau\Big]v^{-}_{\rho}(\boldsymbol{k},t-\tau), \quad (3.60)$$

where $\tau = t - t'$.

In order to evaluate the time integral in equation (3.60), the second boundary-layer-type approximation is now made, which is based on the assumption that the time evolution of low-k modes is much smaller than the time scale of high-k modes (i.e. Markovian approximation). This can be done by expanding $v^{-}_{\rho}(\boldsymbol{k},t-\tau)$ in a Taylor series at $\tau = 0$:

$$v^{-}_{\rho}(\boldsymbol{k},t-\tau) = v^{-}_{\rho}(\boldsymbol{k},t) - \tau\frac{\partial}{\partial s}v^{-}_{\rho}(\boldsymbol{k},s)\Big|_{s=t} + \mathcal{O}\{\tau^2\} \quad (3.61)$$

and truncating at zeroth-order viz.

$$\frac{\partial}{\partial t}v^{-}_{\rho}(\boldsymbol{k},t) \sim 0. \quad (3.62)$$

Thus, equation (3.59) can be written in form

$$\langle u^{+(0)}_{\beta}(\boldsymbol{j},t)u^{+(1)}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c = \frac{2M^{+}_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})D_{\beta\sigma}(\boldsymbol{j})Q^{+}_{v}(j)}{\nu_0 j^2 + \nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}v^{-}_{\rho}(\boldsymbol{k},t). \quad (3.63)$$

3.4.4 Effective viscosity

With all the previous points in mind, the equation for the low-k velocity mode can be written as

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^-(\boldsymbol{k},t) = \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t) + M^-_{\alpha\beta\gamma}(\boldsymbol{k})\int^- d^3j\; u_\beta^-(\boldsymbol{j},t)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t) + \int^+ d^3j\; \frac{4M^-_{\alpha\beta\gamma}(\boldsymbol{k})M^+_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})D_{\beta\sigma}(\boldsymbol{j})Q_v^+(j)}{\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}u_\rho^-(\boldsymbol{k},t), \tag{3.64}$$

where we have now set $\lambda = 1$. we have also made the approximation of putting $v_\alpha^-(\boldsymbol{k},t) \simeq u_\alpha^-(\boldsymbol{k},t)$: note the discussions in section 3.3.

Now, the last term in the RHS of equation (3.64) is linear in the low-k velocity mode, $\boldsymbol{u}^-$, and may be interpreted in terms of an increment to the molecular viscosity, ν_0.

For isotropic field, it is shown [64] that

$$T_{\alpha\rho}(\boldsymbol{k})u_\rho^-(\boldsymbol{k},t) = \frac{1}{d-1}\mathrm{Tr}[T_{\alpha\rho}(\boldsymbol{k})u_\alpha^-(\boldsymbol{k},t)] \tag{3.65}$$

for a d-dimensional system, where

$$T_{\alpha\rho}(\boldsymbol{k}) = \int^+ d^3j\; \frac{4M^-_{\alpha\beta\gamma}(\boldsymbol{k})M^+_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})D_{\beta\sigma}(\boldsymbol{j})Q_v^+(j)}{\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}. \tag{3.66}$$

This allows us to have a rescaled Navier-Stokes equation which looks like the original one, viz.,

$$\left[\frac{\partial}{\partial t}+\nu_1(k)k^2\right] u_\alpha^-(\boldsymbol{k},t) = \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t) + M^-_{\alpha\beta\gamma}(\boldsymbol{k})\int^- d^3j\; u_\beta^-(\boldsymbol{j},t)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t), \tag{3.67}$$

where

$$\nu_1(k) = \nu_0 + \delta\nu_0(k) \tag{3.68}$$

and

$$\delta\nu_0(k) = -\frac{1}{k^2}\int^+ d^3j\; \frac{\frac{4}{d-1}\mathrm{Tr}[M^-_{\alpha\beta\gamma}(\boldsymbol{k})M^+_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})D_{\beta\sigma}(\boldsymbol{j})]}{\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}Q_v^+(j). \tag{3.69}$$

Here, we consider space dimension $d = 3$, and also note that although we have a negative sign on the front of the RHS of the equation (3.69), the value of $\delta\nu_0(k)$ is positive (the detail can be found in the next chapter).

3.5 Concluding remarks

In this chapter, the procedure of the two-field theory proposed by McComb and Watt [51] is recast in terms of the representative model field. The relevant techniques: the perturbation expansion, the operation of the conditional average and the boundary-layer-type approximations are then employed. It has been shown that the result is identical with that previously inferred from the two-field theory: the equation for the high-k modes, (3.67), which has the form invariance under the RG transformation with the expression for the effective viscosity as given by equation (3.68).

At the beginning of this chapter, we state two questions arising from the two-field theory [51] about the expression of the $\boldsymbol{v}^+$ field and the inconsistency of the procedure. In two-field theory, it was assumed that the $\boldsymbol{v}^+$ is a first order Taylor expansion of the $\boldsymbol{u}^+$ field about $k = k_0$: equation (2.13) i.e.

$$u_\alpha^+(\boldsymbol{k},t) = \underbrace{u_\alpha^+(\boldsymbol{k}_0,t) + (\boldsymbol{k}-\boldsymbol{k}_0)\cdot\boldsymbol{\nabla}_k u_\alpha^+(\boldsymbol{k},t)|_{k=k_0}}_{v_\alpha^+(\boldsymbol{k},t)} + \underbrace{\mathcal{O}\{|\boldsymbol{k}-\boldsymbol{k}_0|^2\}}_{\Delta_\alpha^+(\boldsymbol{k},t)}, \qquad (3.70)$$

However, the use of Taylor expansion in connection with chaotic trajectories has been criticized (see section 2.3). Instead, we introduce the explicit expression of the $\boldsymbol{v}^+$ field, equation (3.27), which has the form:

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] v_\alpha^+(\boldsymbol{k},t)$$

$$= M_{\alpha\beta\gamma}^+(\boldsymbol{k}) \int^+ d^3j \left[v_\beta^-(\boldsymbol{j},t) v_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t) + v_\beta^+(\boldsymbol{j},t) v_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t)\right]. \qquad (3.71)$$

Comparing with the Navier-Stokes equation given by (3.12), the absence of explicit coupling terms between the high-k and low-k modes of the $\boldsymbol{v}$ field should

be noted. The ansatz for (3.71) is based on the localness[1] of mode-mode coupling interaction in k-space within the cascade picture of turbulence. In other words, our assumption is in effect that the coupling term we have neglected will not play a major role in the concerns of the energy transfer.

In his paper, Eyink [65] studied the conjecture of Onsager for a solution of the hydrodynamics of incompressible Euler equations. This work concludes that energy transfer is dominated by local triad[2] interactions. In the study of the Navier-Stokes equation, recent numerical simulations [66, 67] have showed that the energy within high wave-number dominantly moves from higher to lower wave-number modes via local triad interactions, whereas the phase information is transferred within more distant triad interactions. It might support our assumption made in section 3.3.2 that, for the model field described in equation (3.71), the phase information between high and low wave-number modes could be neglected by suppressing the coupling term between them, while the model field could represent the correct picture of energy transfer. Then the conditional average can be evaluated exactly for our model field, which is the prime property that makes this model soluble. At this stage, however, the model presented here is preliminary in nature, and work remains to be done. This is currently the subject of our studies.

The other concern about the two-field theory was that the theory was truncated up-to order of $\boldsymbol{\Delta}^+$ described in equation (3.70), and hence the calculations were dependent on performing the two-field decomposition in the low-k equation or the high-k equation. However, our calculations have been based on the perturbation

[1] In the study of turbulence, 'localness' usually refers to scales, not to positions as in other areas of physics.

[2] 'triad' is the group of three wavenumber arguments of velocity field modes. For the case of the Navier-Stokes velocity field described by (3.12), the triad indicates the wavenumber arguments: $\boldsymbol{k}$, $\boldsymbol{j}$ and $\boldsymbol{k}-\boldsymbol{j}$.

expansion of the $\boldsymbol{u}^+$ field, which has the form:

$$u_\alpha^+(\boldsymbol{k},t) = \underbrace{u_\alpha^{+(0)}(\boldsymbol{k},t)}_{v_\alpha^+(\boldsymbol{k},t)} + \underbrace{\lambda u_\alpha^{+(1)}(\boldsymbol{k},t) + \lambda^2 u_\alpha^{+(2)}(\boldsymbol{k},t) + \cdots}_{\Delta_\alpha^+(\boldsymbol{k},t)}, \tag{3.72}$$

and the theory has been truncated up-to order of λ^2 (see the previous section), and thus our calculations are *not* dependent on performing the decomposition, the velocity field $\boldsymbol{u}^+$ to the model field $\boldsymbol{v}^+$ and the correction term $\boldsymbol{\Delta}^+$, at any stage.

Finally, it should be noted here that we have also neglected $\langle \boldsymbol{\Phi}^- \boldsymbol{\Phi}^- \rangle_c$ and higher order terms as small quantities (see section 3.3). For example, the equation for the high wave-number modes after the first shell elimination, equation (3.67), is actually

$$\begin{aligned} \left[\frac{\partial}{\partial t} + \nu_1(k)k^2\right] u_\alpha^-(\boldsymbol{k},t) \;\; &= \;\; \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t) + M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int^- d^3j \; u_\beta^-(\boldsymbol{j},t) u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t) \\ &\quad + \mathcal{E}\{\langle \boldsymbol{\Phi}^- \boldsymbol{\Phi}^- \rangle_c\}. \end{aligned} \tag{3.73}$$

This point is closely related to triple non-linear contributions in our calculations, and will be dealt with in the next chapter.

Chapter 4

Results and Discussions of The Iterative-Averaging Renormalization Group Method

4.1 Introduction

Firstly, in this chapter, we show that the results derived in chapter 3 for the first shell can be extended to further shells. After that, we outline how the theory might be used to evaluate a value for the Kolmogorov constant α in the study of fluid turbulence. Note that the numerical calculations and the code are based on the previous work done by McComb and Watt [62].

Secondly, we discuss the approximation made in our theory: the Markovian approximation in section 3.4.3. In order to eliminate the effects of high wave-number modes ($\boldsymbol{u}^{+}$) on the equation for low wave-number modes ($\boldsymbol{u}^{-}$), we break away from the straightforward step of the direct substitution of $\boldsymbol{u}^{+(1)}$, and instead form an evolution equation for $\boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(1)}$. We argue here that this is a necessary step to carry on further calculations.

We finally discuss triple non-linear contribution (i.e. proportional to $\boldsymbol{uuu}$) in the procedure of renormalization group methods for the Navier-Stokes equation. This is important in terms of maintaining the form invariance of the equation

of motion under renormalization group transformation, and therefore we show details encountered with triple non-linear contributions in our theory. A brief comparison is then made with other methods that previously exist.

4.2 Calculations and results

4.2.1 Further shell eliminations

In chapter 3, we derived an expression of the equation for the low-k velocity mode after the first shell elimination, which has the form

$$\left[\frac{\partial}{\partial t}+\nu_1(k)k^2\right] u_\alpha^-(\boldsymbol{k},t) = \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t)+M^-_{\alpha\beta\gamma}(\boldsymbol{k})\int^- d^3j\; u_\beta^-(\boldsymbol{j},t)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t), \quad (4.1)$$

where

$$\nu_1(k) = \nu_0 + \delta\nu_0(k) \quad (4.2)$$

and the formula for the viscosity increment is

$$\delta\nu_0(k) = \frac{1}{k^2}\int^+ d^3j\; \frac{L(\boldsymbol{k},\boldsymbol{j})Q_v^+(j)}{\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}, \quad (4.3)$$

where the coefficient $L(\boldsymbol{k},\boldsymbol{j})$ in equation (4.3) is given by

$$\begin{aligned} L(\boldsymbol{k},\boldsymbol{j}) &= -2M^-_{\rho\beta\gamma}(\boldsymbol{k})M^+_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})D_{\beta\sigma}(\boldsymbol{j}) \\ &= \frac{(1-\mu^2)[kj(j^2-2k^2)\mu+k^4]}{k^2+j^2-2kj\mu}, \end{aligned} \quad (4.4)$$

where μ is the cosine of the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{j}$. Note that equations (4.3) and (4.4) are different from the earlier forms[1] used in the two-field theory [51].

Now, in order to evaluate the numerical value of the increment to the viscosity described in equation (4.3), we must have an ansatz to relate Q_v^+ to known

[1] The coefficient, for example, $L(\boldsymbol{k},\boldsymbol{j})$ in reference [51] was

$$L(\boldsymbol{k},\boldsymbol{j}) = -2M^-_{\alpha\beta\gamma}(\boldsymbol{k})M^+_{\beta\rho\sigma}(\boldsymbol{j})D_{\gamma\sigma}(\boldsymbol{k}-\boldsymbol{j}) = \frac{[\mu(k^2+j^2)-kj(1+2\mu^2)](1-\mu^2)kj}{k^2+j^2-2kj\mu}. \quad (4.5)$$

However, the result should be same due to a symmetry of the second term of the RHS of equation (4.1) under interchange of the vectors $\boldsymbol{j}$ and $\boldsymbol{k}-\boldsymbol{j}$.

quantities. As mentioned in section 2.3.2, McComb and Watt [51] took $\boldsymbol{v}^+$ to be truncations of the Taylor expansion, to first order, of the $\boldsymbol{u}^+$ about $k = k_0$. Perhaps, it might be true that we cannot expect a good approximation from the expansion of the velocity field in Taylor series due to its chaotic trajectory. However, having specified the model field without making an assumption of the Taylor expansion in the chapter 3, it is our conclusion that the Taylor series can be carried out for the spectral density function Q^+, which has the analytical behavior, about $k = k_0$, thus:

$$\begin{aligned} Q_v^+(k) &= Q^+(k)|_{k=k_0} + (k-k_0)\left.\frac{\partial Q^+(k)}{\partial k}\right|_{k=k_0} + \\ &\quad + \frac{1}{2}(k-k_0)^2\left.\frac{\partial^2 Q^+(k)}{\partial k^2}\right|_{k=k_0} + \text{higher-order terms.} \end{aligned} \tag{4.6}$$

Note that the spectral density function has been given by equation (3.2) and we shall replace it by the Kolmogorov form for the energy spectrum later. Therefore, the spectral density function has the form: $Q(k) \sim k^{-11/3}$.

Since we are using

$$k - k_0 \leq \eta \tag{4.7}$$

as a Taylor expansion parameter, a representation of equation (4.6) will simultaneously face two constraints. Firstly, the bandwidth parameter η must be large enough for the high-k field to be statistically independent of the low-k field. Secondly, η must be small enough to represent Q_v^+ by means of the truncated Taylor series of Q^+. As a consequence of two constraints, we are in fact imposing here both upper and lower bounds on acceptable values of η.

We now have a complete formula for the increment to the viscosity for the first shell up-to the second-order truncation for the spectral density function, Q_v^+, in equation (4.6):

$$\begin{aligned} \delta\nu_0(k) &= \frac{1}{k^2}\int^+ d^3j\, \frac{L(\boldsymbol{k},\boldsymbol{j})}{\nu_0 j^2 + \nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}\left\{Q^+(j)|_{j=k_0} + \right. \\ &\quad \left. + (j-k_0)\left.\frac{\partial Q^+(j)}{\partial j}\right|_{j=k_0} + \frac{1}{2}(j-k_0)^2\left.\frac{\partial^2 Q^+(j)}{\partial j^2}\right|_{j=k_0}\right\}. \end{aligned} \tag{4.8}$$

We, thus, are in position to extend the first shell results to further shells, as follows:

(i) write equation (4.1) by setting $\boldsymbol{u}^- = \boldsymbol{u}$, which is a new Navier-Stokes equation with effective viscosity $\nu_1(k)$ on interval $0 \leq k \leq k_1$, where the wave-number k_1 is now defined by the renormalized dissipation relation [see equation (3.6)]

$$\varepsilon = \int_0^{k_1} dk\ 2\nu_1(k)k^2E(k), \tag{4.9}$$

which can be obtained by forming an energy equation for equation (4.1) [see also section 1.2.3];

(ii) decompose the velocity field $\boldsymbol{u}$ into $\boldsymbol{u}^-$ and $\boldsymbol{u}^+$ at the cut-off wave-number k_2, which has the form

$$k_2 = (1-\eta)k_1 = (1-\eta)^2k_0; \tag{4.10}$$

(iii) repeat the procedures in section 3.3, section 3.4 and this section, in order to eliminate the shell, this time $k_2 \leq k \leq k_1$.

By repeating the calculations, we can have a scaled Navier-Stokes equation for nth shell as

$$\left[\frac{\partial}{\partial t} + \nu_n(k)k^2\right] u_\alpha^-(\boldsymbol{k},t) = \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t) + M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int_{j<k_n} d^3j\ u_\beta^-(\boldsymbol{j},t)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t), \tag{4.11}$$

where the effective viscosity is given by

$$\nu_n(k) = \nu_{n-1}(k) + \delta\nu_{n-1}(k) \tag{4.12}$$

with

$$\begin{aligned}\delta\nu_{n-1}(k) \;=\; & \frac{1}{k^2}\int_{k_n<j<k_{n-1}} d^3j\ \frac{L(\boldsymbol{k},\boldsymbol{j})}{\nu_{n-1}(j)j^2 + \nu_{n-1}(|\boldsymbol{k}-\boldsymbol{j}|)|\boldsymbol{k}-\boldsymbol{j}|^2}\left\{Q^+(j)|_{j=k_{n-1}}\right. \\ & \left. +(j-k_{n-1})\left.\frac{\partial Q^+(j)}{\partial j}\right|_{j=k_{n-1}} + \frac{1}{2}(j-k_{n-1})^2\left.\frac{\partial^2 Q^+(j)}{\partial j^2}\right|_{j=k_{n-1}}\right\}.\end{aligned} \tag{4.13}$$

Here the cut-off wave-number is

$$k_n = (1 - \eta)^n k_0. \tag{4.14}$$

4.2.2 Rescaling of the variables: the evaluation of Kolmogorov constant

In this section, we shall attempt to evaluate a value for the Kolmogorov constant from the results of our theory, by assuming that the energy spectrum $E(k)$ is given by a power law of the form

$$E(k) = \alpha \varepsilon^r k^s. \tag{4.15}$$

In order to do so, we are now dealing with the second part of the RG transformation: that is re-scaling of variables after each shell elimination to make the new equation on the new grid looks as much as possible like the old equation on the old grid. We must do this until the scaled equation eventually reaches the fixed point under RG transformation.

As the only part of the Navier-Stokes equation which changes its form under the transformation is the viscous term, it is necessary to consider the re-scaling of the viscosity, equations (4.12) and (4.13).

We start with the second moment of the turbulent velocity defined in section 1.2.2:

$$\langle u_\alpha(\boldsymbol{k}, t) u_\beta(\boldsymbol{k}', t) \rangle = Q(k) D_{\alpha\beta}(\boldsymbol{k}) \delta(\boldsymbol{k} + \boldsymbol{k}'), \tag{4.16}$$

where

$$E(k) = 4\pi^2 Q(k). \tag{4.17}$$

Invoking that $E(k)$ has the form given in equation (4.15), and hence $Q(k)$ in equation (4.16) goes as a power of k, we simply have to make the scaling transformation for the wave-number, in order to re-scale the viscous term, such as

$$k = k_{n+1} k', \tag{4.18}$$

where k' is a dimensionless variable, and

$$k_{n+1} = hk_n \tag{4.19}$$

with $h = 1 - \eta$.

It has been previously shown by McComb and Watt [51] that the exponents for the energy spectrum and effective viscosity can be fixed by a combination of dimensional analysis and power counting. As a result, the exponents for equation (4.15) are $r = 2/3$ and $s = -5/3$, which is the Kolmogorov energy spectrum. Following McComb and Watt, the effective viscosity can be written in terms of the dimensionless function $\tilde{\nu}_n$ as

$$\nu_n(k'k_n) = \alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(k'). \tag{4.20}$$

Then, the recursion relations (4.2) and (4.13) can be scaled as

$$\tilde{\nu}_n(k') = h^{4/3}\tilde{\nu}_{n-1}(hk'^{1/3}) + h^{-4/3}\delta\tilde{\nu}_{n-1}(k'), \tag{4.21}$$

and

$$\begin{aligned}\delta\tilde{\nu}_n(k') &= \frac{1}{k'^2}\int_{1<j'<1/h} d^3j' \, \frac{L(\boldsymbol{k}', \boldsymbol{j}')}{\tilde{\nu}_{n-1}(hj')j'^2 + \tilde{\nu}_{n-1}(hp')p'^2}\Bigg\{h^{11/3} + \\ &\quad -\frac{11}{3}h^{14/3}(j' - h^{-1}) + \frac{77}{9}h^{17/3}(j' - h^{-1})^2.\Bigg\},\end{aligned} \tag{4.22}$$

where $p' = |\boldsymbol{k}' - \boldsymbol{j}'|$.

We now use the renormalized version of the dissipation relation for the nth shell [see equation (4.9)]:

$$\varepsilon = \int_0^{k_n} dk \; 2\nu_n(k)k^2E(k). \tag{4.23}$$

As equations (4.21) and (4.22) reach a fixed point, we can obtain the Kolmogorov constant from equation (4.23):

$$\alpha = \left[2\int_0^1 dk \; \nu^*(k')k'^{1/3}\right]^{-2/3}, \tag{4.24}$$

where $\nu^*(k')$ indicates the fixed-point viscosity.

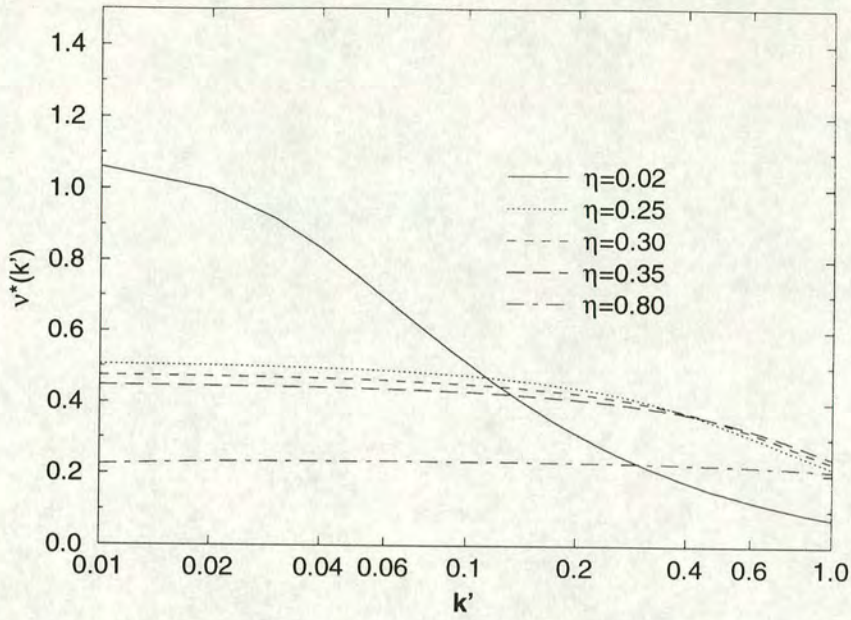

Figure 4.1: Dependence of the fixed-point viscosity based on the $\nu^*(k')$ on wavenumber k', for different values of the bandwidth parameter η.

Equations (4.21), (4.22) and (4.24) have been calculated numerically and the calculation for each value of the bandwidth parameter, η, has reached a fixed point under numerical the iteration of the recursion relation. These calculations were previously reported [51], and therefore we present selected results only in this section.

In figure 4.1, we show the profiles of the fixed-point viscosity $\nu^*(k')$ for several values of the bandwidth parameter η. The calculations are based on the first-order truncation of the Taylor series in the definition of Q_v^+ in equation (4.6). Obviously, the graph show that the fixed-point viscosity $\nu^*(k')$ is dependent on the bandwidth parameter, η. As mentioned in the previous section, the calculation was expected to breakdown for both small and large values of η due to the approximation taken for Q_v^+. However, between the two extreme values of η ($\eta = 0.02$ and 0.80), the shape of the fixed-point viscosity for values of the bandwidth parameter $\eta = 0.25$, 0.30 and 0.35 has a large plateau region for small

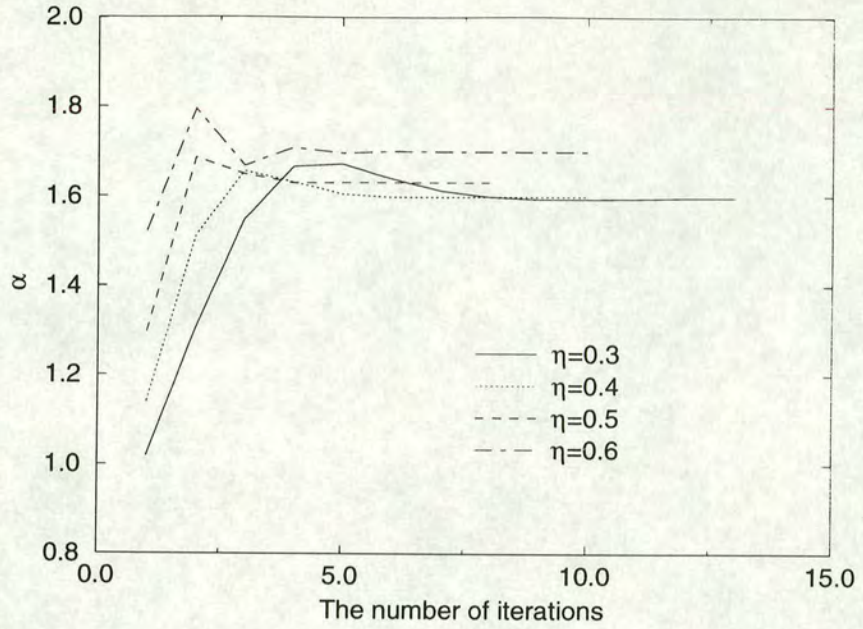

Figure 4.2: Convergence of the Kolmogorov spectral constant α to the fixed points for several values of the bandwidth parameter η.

wave-number modes and an independent behavior of η, where we expect a good estimate for the Kolmogorov constant.

Figure 4.2 shows details of the convergence of the Kolmogorov constant, α, after eliminating each shell for selected values of η. The calculations are based on the first-order truncation of the Taylor series in the definition of Q_v^+ in equation (4.6).

In figure 4.3, we show the results for the Kolmogorov constant α as a function of the bandwidth parameter η. Curves are shown for zeroth-, first- and second-order truncations of the Taylor expansion for Q_v^+ in equation (4.6). It should be noted here that η is not exactly a free parameter, as our calculations diverge at both very small and very large values of η. Evidently, the graph gives a guide to the range of η values for which the theory seems to be performing best. If it was exact, the theory should give a value of α which is independent of η. A plateau

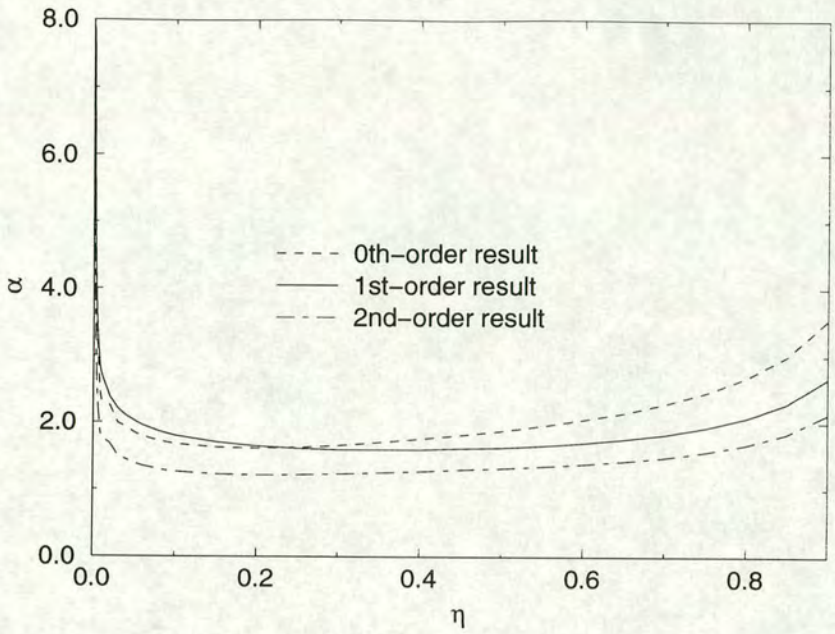

Figure 4.3: Dependence on the bandwidth parameter η of the calculated values of the Kolmogorov spectral constant α based on equation (4.13).

region in which this is approximately true can be seen on all three graphs. The first-order results give a value for α of 1.60 ± 0.01 in the range $\eta = 0.25 - 0.45$. For zeroth-order, we have $\alpha = 1.62 \pm 0.01$ for $\eta = 0.15 - 0.25$, and for second-order $\alpha = 1.22 \pm 0.01$ in the range $\eta = 0.15 - 0.30$. These values compare well with experimental values, as given in a recent survey of many different experiments as $\alpha = 1.619 \pm 0.055$ [9]. An interesting feature of these results is that the first-order curve has a larger plateau region than the zeroth- and second-order, supporting the assumption that the first-order should be the better approximation.

4.3 An investigation of the Markovian approximation

As discussed in section 2.3.2, the two-field theory presented by McComb and Watt [51], suffers from ambiguities. The purpose of the work given in this thesis is to eliminate those ambiguities by introducing the model field and hence the

perturbation theory. The question now considered is: does the present work avoid all the ambiguities? The obvious candidate for a corresponding ambiguity in the present work could be where we depart from the straightforward procedure of the direct substitution of $\boldsymbol{u}^{+(1)}$, and instead form an evolution equation for $\boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(1)}$ in section 3.4.3. In this section, we will argue that the latter procedure is a correct approach in order to make the Markovian approximation properly.

Here, repeat the steps in section 3.4.3 for the term (3.51)

$$2\lambda M^{-}_{\alpha\beta\gamma}(\boldsymbol{k}) \int^{-} d^3j \, \langle u^{+(0)}_{\beta}(\boldsymbol{j},t) u^{+(1)}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c, \tag{4.25}$$

but this time we simply replace $\boldsymbol{u}^{+(1)}$ in (4.25) directly with the $\boldsymbol{u}^{+(1)}$ equation, (3.43). This procedure yields

$$\begin{aligned} u^{+(0)}_{\beta}(\boldsymbol{j},t) u^{+(1)}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t) \;=\; & M^{+}_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j}) \int^{+} d^3p \int_{-\infty}^{t} dt' \, G^{(0)}(|\boldsymbol{k}-\boldsymbol{j}|;t,t') \\ & \times \Big[2u^{+(0)}_{\beta}(\boldsymbol{j},t) u^{-}_{\rho}(\boldsymbol{p},t') u^{-}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t') \\ & - u^{+(0)}_{\beta}(\boldsymbol{j},t) v^{-}_{\rho}(\boldsymbol{p},t') v^{-}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t') \\ & + 2u^{+(0)}_{\beta}(\boldsymbol{j},t) u^{-}_{\rho}(\boldsymbol{p},t') u^{+(0)}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t') \Big]. \end{aligned} \tag{4.26}$$

Again, replacing each $\boldsymbol{u}^{-}$ with $\boldsymbol{v}^{-} + \boldsymbol{\Phi}^{-}$ in equation (4.26):

$$\begin{aligned} u^{+(0)}_{\beta}(\boldsymbol{j},t) u^{+(1)}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t) \;=\; & M^{+}_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j}) \int^{+} d^3p \int_{-\infty}^{t} dt' \, G^{(0)}(|\boldsymbol{k}-\boldsymbol{j}|;t,t') \\ & \times \Big[2u^{+(0)}_{\beta}(\boldsymbol{j},t) v^{-}_{\rho}(\boldsymbol{p},t') v^{-}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t') \\ & + 4u^{+(0)}_{\beta}(\boldsymbol{j},t) \Phi^{-}_{\rho}(\boldsymbol{p},t') v^{-}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t') \\ & + 2u^{+(0)}_{\beta}(\boldsymbol{j},t) \Phi^{-}_{\rho}(\boldsymbol{p},t') \Phi^{-}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t') \\ & - u^{+(0)}_{\beta}(\boldsymbol{j},t) v^{-}_{\rho}(\boldsymbol{p},t') v^{-}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t') \\ & + 2u^{+(0)}_{\beta}(\boldsymbol{j},t) v^{-}_{\rho}(\boldsymbol{p},t') u^{+(0)}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t') \\ & + 2u^{+(0)}_{\beta}(\boldsymbol{j},t) \Phi^{-}_{\rho}(\boldsymbol{p},t') u^{+(0)}_{\sigma}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t') \Big] \end{aligned} \tag{4.27}$$

and taking the conditional average of each term in equation (4.27) gives

$$\langle u_\beta^{+(0)}(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c = 2M^+_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})\int^+ d^3p \int_{-\infty}^{t} dt'\, G^{(0)}(|\boldsymbol{k}-\boldsymbol{j}|;t,t') \times\langle u_\beta^{+(0)}(\boldsymbol{j},t)u_\sigma^{+(0)}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t')\rangle_c v_\rho^-(\boldsymbol{p},t'). \tag{4.28}$$

Upon using equations (3.57) and (3.58), equation (4.28) can be readily written as

$$\langle u_\beta^{+(0)}(\boldsymbol{j},t)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c = 2M^+_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j})D_{\beta\sigma}(\boldsymbol{j})\int_0^\infty d\tau\, G^{(0)}(|\boldsymbol{k}-\boldsymbol{j}|;t,t-\tau)Q_v^+(\boldsymbol{j},\tau)v_\rho^-(\boldsymbol{k},t-\tau), \tag{4.29}$$

where $\tau = t - t'$.

Now, it can be assumed that the $\boldsymbol{v}^-$ are slowly varying on the time scales of the $\boldsymbol{v}^+$ (i.e. Markovian approximation). In order to do this, we may expand $\boldsymbol{v}^-$ and Q_v^+ in a Taylor series in τ, which are

$$v_\rho^-(\boldsymbol{k},t-\tau) = v_\rho^-(\boldsymbol{k},t) - \tau\frac{\partial}{\partial s}v_\rho^-(\boldsymbol{k},s)\bigg|_{s=t} + \mathcal{O}\{\tau^2\} \tag{4.30}$$

and

$$Q_v^+(\boldsymbol{j},\tau) = Q_v^+(\boldsymbol{j}) + \tau\frac{\partial}{\partial\tau}Q_v^+(\boldsymbol{j},\tau)\bigg|_{\tau=0} + \mathcal{O}\{\tau^2\}. \tag{4.31}$$

Now equation (4.30) can be truncated at zeroth-order (see section 3.4.3), but we cannot neglect any terms in equation (4.31). As a result, performing the time integral (τ- integral) in equation (4.29) becomes very difficult due to lack of information about the term like

$$\tau\frac{\partial}{\partial\tau}Q_v^+(\boldsymbol{j},\tau)\bigg|_{\tau=0}. \tag{4.32}$$

It is therefore our conclusion that an evolution equation for $\boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(1)}$ has to be formed, in order to make the proper Markovian approximation and hence

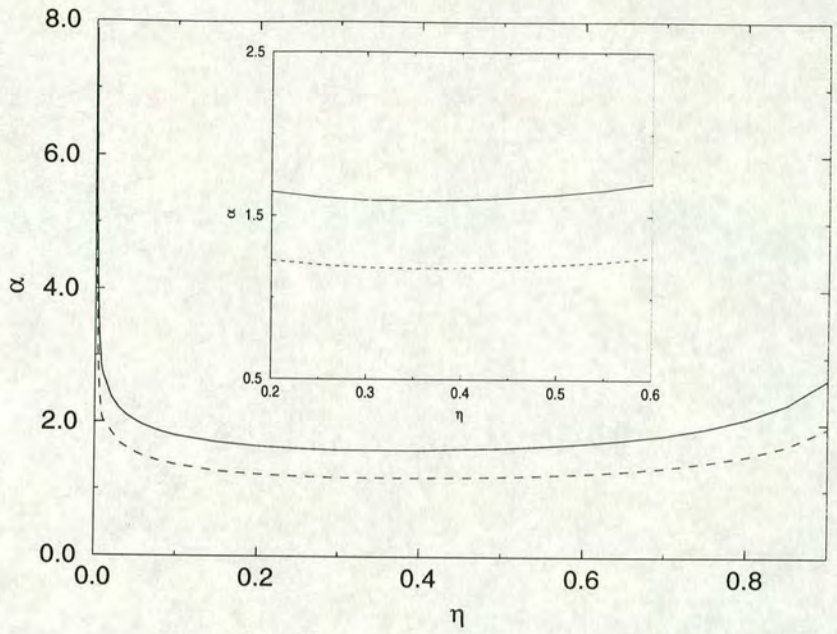

Figure 4.4: Dependence on the bandwidth parameter η of the calculated values of the Kolmogorov spectral constant α based on equation (4.13) for the two lifetimes (solid line) and the one lifetime (broken line).

carry on further calculations. If the approach [the direct substitution of $\boldsymbol{u}^+$ for the term (4.25)] is actually carried out by simply dropping the second and rest terms of equation (4.31), then it would be found that the viscosity increment has a different form from equation (4.13) in having just one inverse lifetime $\nu_0 j^2$ in the denominator. In figure 4.4, we show the effect of calculating the Kolmogorov constant with equation (4.13), which has the two lifetimes and also, for comparison, with the form, based on an uncontrolled approximation, which has one lifetime.

As is so often the case in turbulence theory, the use of uncontrolled approximations does not have a particularly large effect on the answer. However, it was pointed out a long time ago by Edwards [68], who drew an analogy with the Peierls-Boltzmann equation for phonon transport in solids, that a full perturbation solution of the Navier-Stokes equations would involve 'cross-sections' with

three lifetimes i.e.

$$\omega(k) = \nu_0 k^2 + \int d^3j \; \frac{L(\boldsymbol{k},\boldsymbol{j})Q(|\boldsymbol{k}-\boldsymbol{j}|)}{\omega(k)+\omega(j)+\omega(|\boldsymbol{k}-\boldsymbol{j}|)}, \tag{4.33}$$

where $\omega(k) = \nu_0 k^2 + r(k)$, and $r(k)$ represents the effects of the non-linearity. For the kind of mode elimination being discussed here, one of these lifetimes is associated with the explicit scales and hence cannot appear in the expression for modes being eliminated. It is arguable that a form like (4.13), which is quadratic in the interaction strength and involves both the dynamically relevant lifetimes is the simplest possible which can capture all the essential physics.

4.4 The triple non-linear contributions in the iterative-averaging RG method

In the study of Navier-Stokes turbulence by applying RG methods, the appearance of triple non-linearities, especially one of the form $\boldsymbol{u}^-\boldsymbol{u}^-\boldsymbol{u}^-$, has been treated as an obstacle, because it breaks the form invariance of the equation of motion under RG transformation. In other investigations, the triple non-linear contribution has been treated as an irrelevant variable in the limit $k \to 0$ [43]; or as analogous to the third-order moments in the application of RG to the Ising model [57]. However, it is also true that these methods require an arbitrary truncation of $\boldsymbol{u}^-\boldsymbol{u}^-\boldsymbol{u}^-$ to be repeated in the cycle of iteration under RG transformation.

Undoubtedly, one of interesting results of the two-field theory [62] was that an equation for the high wave-number modes does not contain a term $\boldsymbol{u}^-\boldsymbol{u}^-\boldsymbol{u}^-$ and hence preserves the form invariance. However, as stated in section 2.3.2, we have found that the term which gives the viscosity increment appears to be same order of $\boldsymbol{u}^-\boldsymbol{u}^-\boldsymbol{u}^-$, and thus should also be neglected for consistency. In fact, this was not clear, because the theory was truncated up-to order of $\boldsymbol{\Delta}^+$ (i.e. in correspondence with up-to order of λ in our calculations). Instead, upon using the perturbation expansion based on the model field defined in section 3.3, we are now possible to keep the terms up-to order of λ, and this allows one to

see clearly how our equations for the remaining modes [see equation (3.67), for instance] maintains the form invariance after the shell elimination. Therefore, in this section, we briefly outline the treatment of the triple non-linear contributions in our iterative-averaging RG method.

Once we divided the velocity field $\boldsymbol{u}$ into $\boldsymbol{u}^-$ and $\boldsymbol{u}^+$ to carry out RG calculations, there are in fact four possible triple non-linear contributions, each of which breaks the form invariance: $\boldsymbol{u}^+\boldsymbol{u}^+\boldsymbol{u}^+$, $\boldsymbol{u}^-\boldsymbol{u}^+\boldsymbol{u}^+$, $\boldsymbol{u}^-\boldsymbol{u}^-\boldsymbol{u}^+$ and $\boldsymbol{u}^-\boldsymbol{u}^-\boldsymbol{u}^-$. Replacing each $\boldsymbol{u}^-$ with $\boldsymbol{v}^- + \boldsymbol{\Phi}^-$ and each $\boldsymbol{u}^+$ with $\boldsymbol{v}^+ + \lambda\boldsymbol{u}^{+(1)}$, and using the rules for the conditional average defined in section 3.3, the triple non-linear terms will have properties as follows:

- $\boldsymbol{u}^+\boldsymbol{u}^+\boldsymbol{u}^+$:

$$\begin{aligned}\langle \boldsymbol{u}^+\boldsymbol{u}^+\boldsymbol{u}^+\rangle_c &= \langle \boldsymbol{v}^+\boldsymbol{v}^+\boldsymbol{v}^+\rangle + 3\lambda\langle \boldsymbol{v}^+\boldsymbol{v}^+\boldsymbol{u}^{+(1)}\rangle_c + 2\lambda^2\langle \boldsymbol{v}^+\boldsymbol{u}^{+(1)}\boldsymbol{u}^{+(1)}\rangle_c \\ &\quad +\lambda^3\langle \boldsymbol{u}^{+(1)}\boldsymbol{u}^{+(1)}\boldsymbol{u}^{+(1)}\rangle_c \\ &= \langle \boldsymbol{v}^+\boldsymbol{v}^+\boldsymbol{v}^+\rangle + 3\lambda\langle \boldsymbol{v}^+\boldsymbol{v}^+\boldsymbol{u}^{+(1)}\rangle_c + \mathcal{E}\{\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c\}. \quad (4.34)\end{aligned}$$

  Due to the homogeneity property of the Navier-Stokes velocity field, the first term in the RHS of (4.34) is zero, when it appeares in the low-k equation (see section 3.4.2), because $M_{\alpha\beta\gamma}\langle v_\alpha^+ v_\beta^+ v_\gamma^+\rangle$ is zero as $M_{\alpha\beta\gamma}(0) = 0$. We also dropped the second term, $\langle \boldsymbol{v}^+\boldsymbol{v}^+\boldsymbol{u}^{+(1)}\rangle_c$, in comparison to $\boldsymbol{v}^-\langle \boldsymbol{v}^+\boldsymbol{v}^+\rangle$, which will appear in (4.35) (the first boundary-layer-type approximation: see section 3.4.3). Thus, the conditional average of $\boldsymbol{u}^+\boldsymbol{u}^+\boldsymbol{u}^+$ is of order of $\mathcal{E}\{\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c\}$, which has been treated as an error of taking the conditional average, and hence neglected as a small quantity.

- $\boldsymbol{u}^-\boldsymbol{u}^+\boldsymbol{u}^+$:

$$\begin{aligned}\langle \boldsymbol{u}^-\boldsymbol{u}^+\boldsymbol{u}^+\rangle_c &= \boldsymbol{v}^-\langle \boldsymbol{v}^+\boldsymbol{v}^+\rangle_c + 2\lambda\Big(\boldsymbol{v}^-\langle \boldsymbol{u}^{+(1)}\boldsymbol{v}^+\rangle_c + \langle \boldsymbol{\Phi}^-\boldsymbol{u}^{+(1)}\boldsymbol{v}^+\rangle_c\Big) \\ &\quad +\lambda^2\Big(\boldsymbol{v}^-\langle \boldsymbol{u}^{+(1)}\boldsymbol{u}^{+(1)}\rangle_c + \langle \boldsymbol{\Phi}^-\boldsymbol{u}^{+(1)}\boldsymbol{u}^{+(1)}\rangle_c\Big) \\ &= \boldsymbol{v}^-\langle \boldsymbol{v}^+\boldsymbol{v}^+\rangle + \mathcal{E}\{\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c\}. \quad (4.35)\end{aligned}$$

The triple moment of the form, $\boldsymbol{v}^-\langle \boldsymbol{v}^+\boldsymbol{v}^+\rangle_c$, has been interpreted in terms of an increment to the kinematic viscosity, ν_0 (see section 3.4.4).

- $\boldsymbol{u}^-\boldsymbol{u}^-\boldsymbol{u}^+$:

$$\begin{aligned}\langle \boldsymbol{u}^-\boldsymbol{u}^-\boldsymbol{u}^+\rangle_c &= \lambda\Big(\boldsymbol{v}^-\boldsymbol{v}^-\langle \boldsymbol{u}^{+(1)}\rangle_c + 2\boldsymbol{v}^-\langle \boldsymbol{\Phi}^-\boldsymbol{u}^{+(1)}\rangle_c + \langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\boldsymbol{u}^{+(1)}\rangle_c\Big) \\ &= \mathcal{E}\{\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c\}. \end{aligned} \tag{4.36}$$

- $\boldsymbol{u}^-\boldsymbol{u}^-\boldsymbol{u}^-$:

$$\begin{aligned}\langle \boldsymbol{u}^-\boldsymbol{u}^-\boldsymbol{u}^-\rangle_c &= \boldsymbol{v}^-\boldsymbol{v}^-\boldsymbol{v}^- + 3\boldsymbol{v}^-\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c + \langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c \\ &= \boldsymbol{v}^-\boldsymbol{v}^-\boldsymbol{v}^- + \mathcal{E}\{\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c\}. \end{aligned} \tag{4.37}$$

Evidently, the term, $\boldsymbol{v}^-\boldsymbol{v}^-\boldsymbol{v}^-$, which will break the form invariance, is *not* simply zero or negligible. This triple non-linear term, however, does not appear in the result of our RG analysis. Thus, we shall give a brief mathematical treatment of the triple non-linear term under the conditional average analysis in the following.

Without going into great detail here, it is clear if the term, $\boldsymbol{v}^-\boldsymbol{v}^-\boldsymbol{v}^-$, can occur in the final equation for the low-k velocity mode, equation (3.64), it will come from the fourth term in the RHS of equation (3.45), which is

$$M^-_{\alpha\beta\gamma}(\boldsymbol{k})\int^- d^3j\; u^-_\beta(\boldsymbol{j},t)u^{+(1)}_\gamma(\boldsymbol{k}-\boldsymbol{j},t). \tag{4.38}$$

By substituting equation (3.43) into (4.38), we have

$$\begin{aligned} & M^-_{\alpha\beta\gamma}(\boldsymbol{k})\int^- d^3j\; u^-_\beta(\boldsymbol{j},t)u^{+(1)}_\gamma(\boldsymbol{k}-\boldsymbol{j},t) \\ &= M^-_{\alpha\beta\gamma}(\boldsymbol{k})\int^- d^3j\; M^+_{\gamma\delta\sigma}(\boldsymbol{k}-\boldsymbol{j})\int^+ d^3p\int_{-\infty}^{t} dt'\; G^{(0)}(|\boldsymbol{k}-\boldsymbol{j}|;t,t') \\ &\quad\times\Big[u^-_\beta(\boldsymbol{j},t)u^-_\delta(\boldsymbol{p},t')u^-_\sigma(\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{j},t') - u^-_\beta(\boldsymbol{j},t)v^-_\delta(\boldsymbol{p},t')v^-_\sigma(\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad+2u^-_\beta(\boldsymbol{j},t)u^-_\delta(\boldsymbol{p},t')u^{+(0)}_\sigma(\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{j},t')\Big]. \end{aligned} \tag{4.39}$$

Replacing each $\boldsymbol{u}^-$ with $\boldsymbol{v}^- + \boldsymbol{\Phi}^-$, and taking the conditional average of each term in equation (4.39) gives

$$
\begin{aligned}
& M^-_{\alpha\beta\gamma}(\boldsymbol{k}) \int^- d^3j \; u^-_\beta(\boldsymbol{j},t) u^{+(1)}_\gamma(\boldsymbol{k}-\boldsymbol{j},t) \\
&= M^-_{\alpha\beta\gamma}(\boldsymbol{k}) \int^- d^3j \; M^+_{\gamma\delta\sigma}(\boldsymbol{k}-\boldsymbol{j}) \int^+ d^3p \int_{-\infty}^{t} dt' \; G^{(0)}(|\boldsymbol{k}-\boldsymbol{j}|;t,t') \\
&\quad \times \Big[\langle \Phi^-_\beta(\boldsymbol{j},t) \Phi^-_\sigma(\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{j},t') \rangle_c v^-_\delta(\boldsymbol{p},t') \\
&\quad + \langle \Phi^-_\beta(\boldsymbol{j},t) \Phi^-_\delta(\boldsymbol{p},t') \Phi^-_\sigma(\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{j},t') \rangle_c \Big],
\end{aligned}
\tag{4.40}
$$

where the triple non-linear contribution, $\boldsymbol{v}^-\boldsymbol{v}^-\boldsymbol{v}^-$, is canceled at this stage. Thus, we have concluded that the conditional average of the term (4.38) is of order of $\mathcal{E}\{\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^- \rangle_c\}$, and hence neglected the triple non-linear contribution as a small quantity.

To sum up, we have shown the treatments of all the triple non-linearities appeared in the procedures of our iterative-averaging RG method. Consequently, it has been found that the leading contribution of those terms is the first term in the RHS of equation (4.35) which gives the viscosity increment, and the rest are of order of $\mathcal{E}\{\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^- \rangle_c\}$.

The effective viscosity derived in section 3.4.4 is of course intended to take account the effects of the eliminated modes upon the equation of motion for the retained modes. In other words, all the effects of the eliminated modes are interpreted in terms of their mean values upon the new equation for the retained velocity modes; and it is shown that the fluctuations, known as an *eddy noise*, are of lowest order $\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^- \rangle_c$. In the study of large-eddy simulations, the effect of an eddy noise is often referred to as a stochastic *backscatter*. This backscatter has been studied theoretically [69, 70], and some of recent large-eddy simulations have included a so-called subgrid stochastic force as a correction of their model [71, 72].

On the other hand, in the study of RG methods applied to fluid turbulence, the

effect of a stochastic backscatter (or an eddy noise) has been typically interpreted in terms of a triple non-linear contribution (e.g. $\boldsymbol{u}^{-}\boldsymbol{u}^{-}\boldsymbol{u}^{-}$ for the Navier-Stokes equation). Rose [48] introduced such an effect as part of a new scalar diffusion equation ($\phi^{-}\boldsymbol{u}^{-}\boldsymbol{u}^{-}$) in his study of the problem of passive scalar transport. For the case of the Navier-Stokes equation, Zhou, Vahala and Hussain [50] suggested a drain viscosity to supplement the triple non-linear contribution. The justification for allowing such an prescription for the effect of the backscatter, however, remains to be explained (see section 2.3.1).

Instead, we have shown in chapter 3 and this section that the leading order of the triple non-linear contributions is the effective viscosity derived in section 3.4.4, and others are neglected as $\mathcal{E}\{\langle\boldsymbol{\Phi}^{-}\boldsymbol{\Phi}^{-}\rangle_c\}$. In this consideration, nonetheless, the further investigation of $\boldsymbol{\Phi}^{-}$ should be carried out.

It might be worth noting again that $\boldsymbol{\Phi}^{-}$ defined in section 3.3.3 is the uncertainty between two realizations, $\boldsymbol{u}^{-}$ and $\boldsymbol{v}^{-}$. Due to the chaotic nature of turbulence, one could expect that any slight discrepancy between two realizations at the cut-off wave-number $k = k_1$ would amplify rapidly, leading to complete statistical independence through much of the band up to the maximum cut-off wave-number $k = k_0$. Therefore, $\boldsymbol{\Phi}^{-}$ could be taken to be small and the invariance of the second moment of $\boldsymbol{u}^{-}$ would hold to the second order in small quantities, i.e.

$$\langle\boldsymbol{u}^{-}\boldsymbol{u}^{-}\rangle_c = \boldsymbol{v}^{-}\boldsymbol{v}^{-} + \langle\boldsymbol{\Phi}^{-}\boldsymbol{\Phi}^{-}\rangle_c \cong \boldsymbol{v}^{-}\boldsymbol{v}^{-}. \tag{4.41}$$

From the above statements, we have neglected $\langle\boldsymbol{\Phi}^{-}\boldsymbol{\Phi}^{-}\rangle_c$ and higher-order terms as errors in the procedure of our iterative-averaging RG method. Consequently, we have derived the equation of motion for the remaining modes that preserves the form invariance under renormalization group transformation. However, in order to estimate the error, $\mathcal{E}\{\langle\boldsymbol{\Phi}^{-}\boldsymbol{\Phi}^{-}\rangle_c\}$, involved in such an assumption, this point needs further attention in terms of quantitative analysis and this is currently under way.

Chapter 5

The Application of The Iterative-Averaging RG Method to Passive Scalar Transport

5.1 Introduction

We now apply the method developed in chapter 3 to the problem of the convection of a passive scalar field by the Navier-Stokes velocity field.

The attempt to apply the RG method to a passive scalar transport was first introduced by Rose [48], who adapted a straightforward iterative perturbation method. His method, however, involved an uncontrolled approximation to eliminate the effect of the high wave-number modes on the equation for the low wave-number, i.e. the usage of the filtered average over the velocity field (see section 2.3.1). The method also required an arbitrary truncation for a new nonlinear term $\phi \boldsymbol{u}\boldsymbol{u}$, which he treated as a part of a new scalar diffusion equation, in order to maintain the form invariance of the equation of motion under RG transformation.

Later, the idea of the two-field theory [51] applied to the Navier-Stokes velocity field was extended to the problem of a passive scalar transport [64, 73]. As a result, it has been found that the physical constants, such as the Obukhov-

Corrsin constant and the turbulent Prandtl number, can be obtained. However, the method had same problems, which could be found in the procedure of the two-field theory (see section 2.3.2). Therefore, we attempt to deal with the problems by applying the method developed in chapter 3 to the case of passive scalar transport.

In this chapter, we give an alternative procedure of the calculations previously given in references [64] and [73] by introducing a relevant model field analogous to that in chapter 3. The chapter is set out as follows: in section 2, we present the basic equations for the case of a scalar field convected by the Navier-Stokes velocity field; in section 3, we extend the concept of the conditional average to the scalar field as well as the velocity field; section 4 is a detailed account of the first shell elimination by means of a formal perturbation expansion.

5.2 Basic equations for passive scalar transport

Consider a passive scalar field $\phi(\boldsymbol{x},t)$ advected by a turbulent velocity field $u_\alpha(\boldsymbol{x},t)$:

$$\frac{\partial}{\partial t}\phi(\boldsymbol{x},t)+u_\alpha(\boldsymbol{x},t)\frac{\partial}{\partial x_\alpha}\phi(\boldsymbol{x},t)=\mu_0\boldsymbol{\nabla}^2\phi(\boldsymbol{x},t), \tag{5.1}$$

where μ_0 is the molecular diffusivity of the scalar field and the velocity field $u_\alpha(\boldsymbol{x},t)$ satisfies the Navier-Stokes equation given by equation (1.2). Similar to equation (3.5) for the velocity field, the Fourier transform of (5.1) on interval $0\leq k\leq k_0$ can be written as

$$\left[\frac{\partial}{\partial t}+\mu_0 k^2\right]\phi(\boldsymbol{k},t)=\mathcal{F}^{\ll}(\boldsymbol{k},t)+N_\alpha(\boldsymbol{k})\int_{j<k_0}d^3j\;u_\alpha(\boldsymbol{k}-\boldsymbol{j},t)\phi(\boldsymbol{j},t), \tag{5.2}$$

where

$$N_\alpha(\boldsymbol{k})=-ik_\alpha, \tag{5.3}$$

and k_0 is the cut-off wave-number.

For the scalar field, Corrsin [74] has applied the same type of arguments used in Kolmogorov's hypothesis of the velocity field, and derived a power law for the

scalar variance spectrum $E_\phi(k)$ within the inertial range of wave-number space

$$E_\phi(k) = \beta\varepsilon_\phi\varepsilon^{-1/3}k^{-5/3}, \tag{5.4}$$

where ε_ϕ is the rate of dissipation of scalar variance, and β is the Obukhov-Corrsin constant. The maximum cut-off wave-number, k_ϕ, is then given by the dissipation integral:

$$\varepsilon_\phi = \int_0^\infty dk\ 2\mu_0 E_\phi(k) \simeq \int_0^{k_\phi} dk\ 2\mu_0 E_\phi(k). \tag{5.5}$$

For the purpose of applying the RG method, we first split the scalar fields into two parts:

$$\phi(\boldsymbol{k},t) = \begin{cases} \phi^-(\boldsymbol{k},t) & \text{for} \quad 0 < k < k_1 \\ \phi^+(\boldsymbol{k},t) & \text{for} \quad k_1 < k < k_0, \end{cases} \tag{5.6}$$

where k_0 is of the same order of k_ϕ in (5.5) and the cut-off wave-number $k_1 = (1-\eta)k_0$. Then, the separate filtered equations for low-k and high-k components are:

$$\begin{aligned} \left[\frac{\partial}{\partial t} + \mu_0 k^2\right]\phi^-(\boldsymbol{k},t) &= \mathcal{F}^{\ll}(\boldsymbol{k},t) + N_\alpha^-(\boldsymbol{k})\int^- d^3j\ \Big[u_\alpha^-(\boldsymbol{k}-\boldsymbol{j},t)\phi^-(\boldsymbol{j},t) \\ &\quad + u_\alpha^-(\boldsymbol{k}-\boldsymbol{j},t)\phi^+(\boldsymbol{j},t) + u_\alpha^+(\boldsymbol{k}-\boldsymbol{j},t)\phi^-(\boldsymbol{j},t) \\ &\quad + u_\alpha^+(\boldsymbol{k}-\boldsymbol{j},t)\phi^+(\boldsymbol{j},t)\Big]; \\ \left[\frac{\partial}{\partial t} + \mu_0 k^2\right]\phi^+(\boldsymbol{k},t) &= N_\alpha^+(\boldsymbol{k})\int^+ d^3j\ \Big[u_\alpha^-(\boldsymbol{k}-\boldsymbol{j},t)\phi^-(\boldsymbol{j},t) \\ &\quad + u_\alpha^-(\boldsymbol{k}-\boldsymbol{j},t)\phi^+(\boldsymbol{j},t) + u_\alpha^+(\boldsymbol{k}-\boldsymbol{j},t)\phi^-(\boldsymbol{j},t) \\ &\quad + u_\alpha^+(\boldsymbol{k}-\boldsymbol{j},t)\phi^+(\boldsymbol{j},t)\Big], \end{aligned} \tag{5.7}$$

where the superscripts "$-$" and "$+$" on $N_\alpha(\boldsymbol{k})$ have the same significance for $\phi(\boldsymbol{k},t)$, and the dynamical equations for $u_\alpha^-(\boldsymbol{k},t)$ and $u_\alpha^+(\boldsymbol{k},t)$ are defined in section 3.2.

5.3 Conditional average of the scalar field

Our purpose is now to introduce the operation of taking the conditional average for the passive scalar field. The concept of the conditional average discussed in section 3.1 is now extended to the sub-ensemble of all ϕ and $\boldsymbol{u}$ fields satisfying (5.2) and (3.5) respectively, such that ϕ^- and $\boldsymbol{u}^-$ are approximately constant in all elements of the sub-ensemble. Then the properties of the conditional average involving the low-k modes are:

$$\langle \boldsymbol{u}^- \rangle_c \simeq \boldsymbol{u}^-; \tag{5.8}$$

$$\langle \phi^- \rangle_c \simeq \phi^-. \tag{5.9}$$

Just as in the case of the velocity field, let us introduce a test model field ψ for the passive scalar field, which is obtained by choosing any member of the sub-ensemble set to provide its low-k modes and importantly another unconnected realization to provide its high-k modes. Then, the dynamical equation, which represents the high-k modes of ψ, might be written as:

$$\left[\frac{\partial}{\partial t} + \mu_0 k^2\right] \psi^+(\boldsymbol{k},t)$$

$$= N_\alpha^+(\boldsymbol{k}) \int^+ d^3j \left[v_\alpha^-(\boldsymbol{k}-\boldsymbol{j},t)\psi^-(\boldsymbol{j},t) + v_\alpha^+(\boldsymbol{k}-\boldsymbol{j},t)\psi^+(\boldsymbol{j},t)\right], \tag{5.10}$$

where the $\boldsymbol{v}$ field is the test field for the Navier-Stokes velocity fields given in section 3.2. The ansatz for postulating the form of equation (5.10) is based on the idea, discussed in section 3.2, which was the localness of mode-mode interaction in k-space within the cascade picture of turbulence.

The properties of the test field, ψ, under the conditional average are:

- The ψ^- field is held constant, viz

$$\langle \psi^- \rangle_c = \psi^-; \tag{5.11}$$

- The conditional average of the ψ^+ field is the same as the full ensemble average

$$\langle \psi^+ \rangle_c = \langle \psi^+ \rangle = 0, \tag{5.12}$$

since the ψ field is homogeneous;

- The isotropy of $\boldsymbol{v}$ and ψ fields leads to

$$\langle \boldsymbol{v}^+ \psi^+ \rangle_c = \langle \boldsymbol{v}^+ \psi^+ \rangle = 0. \tag{5.13}$$

As for the velocity field, we can relate ϕ and ψ by writing

$$\begin{aligned} \phi^- &= \psi^- + \zeta^-; \\ \phi^+ &= \psi^+ + \gamma^+. \end{aligned} \tag{5.14}$$

Note that ζ^- and γ^+ are analogous to $\boldsymbol{\Phi}^-$ and $\boldsymbol{\Delta}^+$ respectively for the velocity field. Then, the conditional averages for second-order moments of ϕ are as follows:

$$\begin{aligned} \langle \phi^- \phi^- \rangle_c &= \psi^- \psi^- + \langle \zeta^- \zeta^- \rangle_c \simeq \psi^- \psi^-; \\ \langle \phi^- \phi^+ \rangle_c &= \psi^- \langle \gamma^+ \rangle_c + \langle \zeta^- \gamma^+ \rangle_c; \\ \langle \phi^+ \phi^+ \rangle_c &= \langle \psi^+ \psi^+ \rangle + 2 \langle \psi^+ \gamma^+ \rangle_c + \langle \gamma^+ \gamma^+ \rangle_c. \end{aligned} \tag{5.15}$$

It should be noted that the term $\langle \zeta^- \zeta^- \rangle_c$ and higher-order terms will be neglected and treated as errors of taking the conditional average operation (see also section 3.3).

Evidently, our problem is now knowing how to relate the conditional averages involving γ^+ to the full ensemble average. This will be the subject of the next section.

5.4 Procedure: first shell elimination

5.4.1 High wave-number equation: double expansion

The purpose of this section is to give a formulation to handle γ^+ and $\boldsymbol{\Delta}^+$ in detail, in terms of the perturbation expansion. Due to the presence of the terms in ϕ^+ and $\boldsymbol{u}^+$ on equation (5.7), the solution for the passive scalar transport requires double expansion, i.e.:

$$\phi^+ = \phi^{+(0)} + \lambda_1\phi^{+(1)} + \lambda_1^2\phi^{+(2)} + \cdots; \tag{5.16}$$

$$\boldsymbol{u}^+ = \boldsymbol{u}^{+(0)} + \lambda_2\boldsymbol{u}^{+(1)} + \lambda_2^2\boldsymbol{u}^{+(2)} + \cdots, \tag{5.17}$$

where λ_1 and λ_2 are the book-keeping parameters and will be set equal to unity at the end of the calculations.

First, subtracting equation (5.10) from equation (5.7) gives

$$\begin{aligned}\phi^+(\boldsymbol{k},t) = {} & \psi^+(\boldsymbol{k},t) + \lambda_1 N_\alpha^+(\boldsymbol{k}) \int^+ d^3j \int_{-\infty}^{t} dt\; G_\phi^{(0)}(k;,t,t') \\ & \times\Big[u_\alpha^-(\boldsymbol{j},t')\phi^-(\boldsymbol{k}-\boldsymbol{j},t') + u_\alpha^-(\boldsymbol{j},t')\phi^+(\boldsymbol{k}-\boldsymbol{j},t') \\ & + u_\alpha^+(\boldsymbol{j},t')\phi^-(\boldsymbol{k}-\boldsymbol{j},t') + u_\alpha^+(\boldsymbol{j},t')\phi^+(\boldsymbol{k}-\boldsymbol{j},t') \\ & - v_\alpha^-(\boldsymbol{j},t')\psi^-(\boldsymbol{k}-\boldsymbol{j},t') - v_\alpha^+(\boldsymbol{j},t')\psi^+(\boldsymbol{k}-\boldsymbol{j},t')\Big],\end{aligned} \tag{5.18}$$

where we have inserted a factor λ_1 and the Green function $G_\phi^{(0)}(k;,t,t')$ is given by

$$\left[\frac{\partial}{\partial t} + \mu_0 k^2\right] G_\phi^{(0)}(k;t,t') = \delta(t-t') \tag{5.19}$$

with the solution

$$G_\phi^{(0)}(k;t,t') = \exp\left[-\mu_0 k^2(t-t')\right]. \tag{5.20}$$

Upon using the zeroth-order solution of equation (5.16) by taking $\phi^{+(0)} = \psi^+$, we can now solve ϕ^+ by means of perturbation expansion, with the results as follows:

$$\phi^{+(0)}(\boldsymbol{k},t) = \psi^{+(0)}(\boldsymbol{k},t); \tag{5.21}$$

$$\begin{aligned}\phi^{+(1)}(\boldsymbol{k},t) &= N_\alpha^+(\boldsymbol{k})\int^+ d^3j \int_{-\infty}^t dt'\, G_\phi^{(0)}(k;t,t')\Big[u_\alpha^-(\boldsymbol{j},t')\phi^-(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad -v_\alpha^-(\boldsymbol{j},t')\psi^-(\boldsymbol{k}-\boldsymbol{j},t') + u_\alpha^-(\boldsymbol{j},t')\phi^{+(0)}(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad +u_\alpha^+(\boldsymbol{j},t')\phi^-(\boldsymbol{k}-\boldsymbol{j},t')\Big]; \qquad (5.22)\end{aligned}$$

$$\begin{aligned}\phi^{+(2)}(\boldsymbol{k},t) &= N_\alpha^+(\boldsymbol{k})\int^+ d^3j \int_{-\infty}^t dt'\, G_\phi^{(0)}(k;t,t')\Big[u_\alpha^-(\boldsymbol{j},t')\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad +u_\alpha^+(\boldsymbol{j},t')\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t')\Big], \qquad (5.23)\end{aligned}$$

and so on.

Obviously, equations (5.22) and (5.23) still require the solution of the Navier-Stokes velocity field. This can be resolved by substituting (5.17) into (5.22) and (5.23) with the solutions defined in section 3.4.1, thus:

$$\begin{aligned}\phi^{+(1)}(\boldsymbol{k},t) &= N_\alpha^+(\boldsymbol{k})\int^+ d^3j \int_{-\infty}^t dt'\, G_\phi^{(0)}(k;t,t')\Big[u_\alpha^-(\boldsymbol{j},t')\phi^-(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad -v_\alpha^-(\boldsymbol{j},t')\psi^-(\boldsymbol{k}-\boldsymbol{j},t') + u_\alpha^-(\boldsymbol{j},t')\phi^{+(0)}(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad +u_\alpha^{+(0)}(\boldsymbol{j},t')\phi^-(\boldsymbol{k}-\boldsymbol{j},t')\Big] \\ &\quad +\lambda_2 N_\alpha^+(\boldsymbol{k})\int^+ d^3j \int_{-\infty}^t dt'\, G_\phi^{(0)}(k;t,t')u_\alpha^{+(1)}(\boldsymbol{j},t')\phi^-(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad +\lambda_2^2 N_\alpha^+(\boldsymbol{k})\int^+ d^3j \int_{-\infty}^t dt'\, G_\phi^{(0)}(k;t,t')u_\alpha^{+(2)}(\boldsymbol{j},t')\phi^-(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad +\mathcal{O}\{\lambda_2^3\}; \qquad (5.24)\end{aligned}$$

$$\begin{aligned}\phi^{+(2)}(\boldsymbol{k},t) &= N_\alpha^+(\boldsymbol{k})\int^+ d^3j \int_{-\infty}^t dt'\, G_\phi^{(0)}(k;t,t')\Big[u_\alpha^-(\boldsymbol{j},t')\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad +u_\alpha^{+(0)}(\boldsymbol{j},t')\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t')\Big] \\ &\quad +\lambda_2 N_\alpha^+(\boldsymbol{k})\int^+ d^3j \int_{-\infty}^t dt'\, G_\phi^{(0)}(k;t,t')u_\alpha^{+(1)}(\boldsymbol{j},t')\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad +\lambda_2^2 N_\alpha^+(\boldsymbol{k})\int^+ d^3j \int_{-\infty}^t dt'\, G_\phi^{(0)}(k;t,t')u_\alpha^{+(2)}(\boldsymbol{j},t')\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t') \\ &\quad +\mathcal{O}\{\lambda_2^3\}. \qquad (5.25)\end{aligned}$$

5.4.2 Low wave-number equation

By substituting the solution of ϕ^+, (5.21) - (5.24), into equation (5.7), we have

$$\begin{aligned}
&\left[\frac{\partial}{\partial t}+\mu_0 k^2\right]\phi^-(\boldsymbol{k},t) \\
&= \mathcal{F}^{\ll}(\boldsymbol{k},t)+N_\alpha^-(\boldsymbol{k})\int^- d^3j\,\left[u_\alpha^-(\boldsymbol{j},t)\phi^-(\boldsymbol{k}-\boldsymbol{j},t)+u_\alpha^-(\boldsymbol{j},t)\phi^{+(0)}(\boldsymbol{k}-\boldsymbol{j},t)\right. \\
&\quad \left.+u_\alpha^{+(0)}(\boldsymbol{j},t)\phi^-(\boldsymbol{k}-\boldsymbol{j},t)+u_\alpha^{+(0)}(\boldsymbol{j},t)\phi^{+(0)}(\boldsymbol{k}-\boldsymbol{j},t)\right] \\
&\quad +\lambda_1 N_\alpha^-(\boldsymbol{k})\int^- d^3j\,\left[u_\alpha^-(\boldsymbol{j},t)\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)+u_\alpha^{+(0)}(\boldsymbol{j},t)\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\right] \\
&\quad +\lambda_2 N_\alpha^-(\boldsymbol{k})\int^- d^3j\,\left[u_\alpha^{+(1)}(\boldsymbol{j},t)\phi^-(\boldsymbol{k}-\boldsymbol{j},t)+u_\alpha^{+(1)}(\boldsymbol{j},t)\phi^{+(0)}(\boldsymbol{k}-\boldsymbol{j},t)\right] \\
&\quad +\mathcal{O}\{\lambda_i^2\}, \qquad (5.26)
\end{aligned}$$

where $i = 1$ or 2.

Then, taking the conditional average of each term of equation (5.26) according to the rules of CA discussed in section 5.3 gives

$$\begin{aligned}
\left[\frac{\partial}{\partial t}+\mu_0 k^2\right]\phi^-(\boldsymbol{k},t) &= \mathcal{F}^{\ll}(\boldsymbol{k},t)+N_\alpha^-(\boldsymbol{k})\int^- d^3j\; u_\alpha^-(\boldsymbol{j},t)\phi^-(\boldsymbol{k}-\boldsymbol{j},t) \\
&\quad +\lambda_1 N_\alpha^-(\boldsymbol{k})\int^- d^3j\;\langle u_\alpha^{+(0)}(\boldsymbol{j},t)\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c \\
&\quad +\mathcal{E}\{\langle\boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c\}+\mathcal{E}\{\langle\zeta^-\zeta^-\rangle_c\}+\mathcal{O}\{\lambda_i^2\}, \qquad (5.27)
\end{aligned}$$

where $\mathcal{E}\{\langle\zeta^-\zeta^-\rangle_c\}$ represents the error terms which contain $\langle\zeta^-\zeta^-\rangle_c$ and higher order, and is analogous to $\mathcal{E}\{\langle\boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c\}$ for the Navier-Stokes velocity field (see section 3.4.2).

5.4.3 Boundary-layer-type approximations

Now consider the second term in the RHS of equation (5.27), which will provide the increment to the diffusivity. It should be noted here that the straightforward approach by directly substituting $\phi^{+(1)}$ leads to uncontrolled approximations (see

section 3.4.3 and 4.2). Alternatively, we form the evolution equation for the term $\langle \boldsymbol{u}^{+(0)}\phi^{+(1)}\rangle_c$ as follows:

(i) rewrite equation (3.27) for $[\partial_t + \nu_0 j^2]u_\alpha^{+(0)}(\boldsymbol{j},t)$ on the LHS and multiply it through by $\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)$: note that $\boldsymbol{u}^{+(0)} \equiv \boldsymbol{v}^+$;

(ii) rewrite equation (5.24) for $[\partial_t + \mu_0|\boldsymbol{k}-\boldsymbol{j}|^2]\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)$ and multiply it by $u_\alpha^{+(0)}(\boldsymbol{j},t)$;

(iii) add the two equations produced by steps (i) and (ii).

This procedure yields

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + \nu_0 j^2 + \mu_0|\boldsymbol{k}-\boldsymbol{j}|^2\right] u_\alpha^{+(0)}(\boldsymbol{j},t)\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t) \\
&= M^+_{\alpha\delta\epsilon}(\boldsymbol{j})\int^+ d^3q\,\Big[v_\alpha^-(\boldsymbol{q},t)v_\epsilon^-(\boldsymbol{j}-\boldsymbol{q},t)\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t) \\
&\quad + u_\alpha^{+(0)}(\boldsymbol{q},t)u_\epsilon^{+(0)}(\boldsymbol{j}-\boldsymbol{q},t)\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\Big] \\
&\quad + N^+_\sigma(\boldsymbol{k}-\boldsymbol{j})\int^+ d^3p\,\Big[u_\sigma^-(\boldsymbol{p},t)\phi^-(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t)u_\alpha^{+(0)}(\boldsymbol{j},t) \\
&\quad - v_\sigma^-(\boldsymbol{p},t)\psi^-(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t)u_\alpha^{+(0)}(\boldsymbol{j},t) \\
&\quad + u_\sigma^-(\boldsymbol{p},t)\phi^{+(0)}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t)u_\alpha^{+(0)}(\boldsymbol{j},t) \\
&\quad + u_\sigma^{+(0)}(\boldsymbol{p},t)\phi^-(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t)u_\alpha^{+(0)}(\boldsymbol{j},t)\Big],
\end{aligned} \tag{5.28}$$

and then taking the conditional average of each term gives

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + \mu_0 j^2 + \nu_0|\boldsymbol{k}-\boldsymbol{j}|^2\right] \langle u_\alpha^{+(0)}(\boldsymbol{j},t)\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c \\
&= M^+_{\alpha\delta\epsilon}(\boldsymbol{j})\int^+ d^3q\,\langle u_\alpha^{+(0)}(\boldsymbol{q},t)u_\epsilon^{+(0)}(\boldsymbol{j}-\boldsymbol{q},t)\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c \\
&\quad + N^+_\sigma(\boldsymbol{k}-\boldsymbol{j})\int^+ d^3p\,\langle u_\sigma^{+(0)}(\boldsymbol{p},t)u_\alpha^{+(0)}(\boldsymbol{j},t)\rangle_c\phi^-(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{p},t).
\end{aligned} \tag{5.29}$$

Now, we take two boundary-layer-style approximations, described in section 3.4.3, for equation (5.29). Firstly, the term $\langle \boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(0)}\phi^{+(1)}\rangle_c$ is ignored in com-

parison to $\langle \boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(0)}\rangle_c\phi^-$. Thus equation (5.29) has the form

$$\langle u_\alpha^{+(0)}(\boldsymbol{j},t)\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c = N_\sigma^+(\boldsymbol{k}-\boldsymbol{j})D_{\sigma\alpha}(\boldsymbol{j})Q_v^+(j)\int_0^\infty d\tau \times \exp\Big[-(\nu_0 j^2+\mu_0|\boldsymbol{k}-\boldsymbol{j}|^2)\tau\Big]\phi^-(\boldsymbol{k},t-\tau), \tag{5.30}$$

where we invoke that $\boldsymbol{u}^{+(0)}$ ($\equiv \boldsymbol{v}^+$) is statistically homogeneous, isotropic and stationary.

At this stage, we take our second type of boundary-layer-style approximation (i.e. Markovian approximation) by assuming $\boldsymbol{u}^{+(0)}$ and $\phi^{+(1)}$. As for the velocity field, we do this by expanding $\phi^-(\boldsymbol{k},t-\tau)$ in a Taylor series in τ about $\tau = 0$ and truncating at zero order. This gives

$$\langle u_\alpha^{+(0)}(\boldsymbol{j},t)\phi^{+(1)}(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c = \frac{N_\sigma^+(\boldsymbol{k}-\boldsymbol{j})D_{\sigma\alpha}(\boldsymbol{j})Q_v^+(j)}{\nu_0 j^2+\mu_0|\boldsymbol{k}-\boldsymbol{j}|^2}\phi^-(\boldsymbol{k},t). \tag{5.31}$$

5.4.4 Results

By substituting equation (5.31) into equation (5.27), we can finally have the equation for the low-k modes of the scalar field as

$$\left[\frac{\partial}{\partial t}+\mu_0 k^2\right]\phi^-(\boldsymbol{k},t) = \mathcal{F}^{\ll}(\boldsymbol{k},t)+N_\alpha^-(\boldsymbol{k})\int^- d^3j\; u_\alpha^-(\boldsymbol{j},t)\phi^-(\boldsymbol{k}-\boldsymbol{j},t) + \int^+ d^3j\;\frac{D_{\sigma\alpha}(\boldsymbol{j})N_\alpha^-(\boldsymbol{k})N_\sigma^+(\boldsymbol{k}-\boldsymbol{j})Q_v^+(j)}{\nu_0 j^2+\mu_0|\boldsymbol{k}-\boldsymbol{j}|^2}\phi^-(\boldsymbol{k},t), \tag{5.32}$$

where we have set $\lambda_1 = \lambda_2 = 1$.

Evidently, the second term in the RHS of equation (5.32) can be interpreted in terms of our increment to the diffusivity, and then we can have a rescaled equation for the passive scalar transport, which looks like the original one:

$$\left[\frac{\partial}{\partial t}+\mu_1(k)k^2\right]\phi^-(\boldsymbol{k},t) = \mathcal{F}^{\ll}(\boldsymbol{k},t)+N_\alpha^-(\boldsymbol{k})\int^- d^3j\; u_\alpha^-(\boldsymbol{j},t)\phi^-(\boldsymbol{k}-\boldsymbol{j},t), \tag{5.33}$$

where

$$\mu_1(k) = \mu_0 + \delta\mu_0(k) \tag{5.34}$$

and

$$\delta\mu_0(k) = \frac{1}{k^2} \int^+ d^3j \; \frac{k_\alpha k_\sigma D_{\sigma\alpha}(\boldsymbol{j}) Q_v^+(j)}{\nu_0 j^2 + \mu_0 |\boldsymbol{k} - \boldsymbol{j}|^2}. \tag{5.35}$$

Here, we recall $N_\alpha(\boldsymbol{k}) = ik_\alpha$ and use the property: $k_\alpha D_{\alpha\beta}(\boldsymbol{k}) = 0$.

5.5 comments

This chapter aims to achieve a systematic re-derivation of the method previously reported in references [64] and [73]. By using the techniques developed in chapter 3, we have derived an equation for the remaining low wave-number modes, and an expression for the effective diffusivity that can be used in the computational calculations to obtain physical measurable.

Those results in the previous section can be extended to further shells, and it has been shown elsewhere [64, 73] that a fixed point is reached under numerical iteration of the recursion relations. Note that we can adapt the methods developed in references [64] and [73] to the detailed calculations after the first shell elimination, since we have derived the identical results: equations (5.33) - (5.35). In order to carry out the calculation, it is necessary to introduce an *ansatz* to relate the ensemble average over the model field to that over the actual field [e.g. Q_v^+ to Q^+: see section 4.1.1]. This leads to a calculation of the Obukhov-Corrsin constant $\beta = 1.02 \pm 0.01$ independent of the bandwidth of modes being eliminated for bandwidths in the range $0.17 \leq \eta \leq 0.33$, in agreement with experiment. It has been found that a range of results for the value β is $0.52 \leq \beta \leq 1.93$ [75], and recently $0.92 \leq \beta \leq 1.53$ [76].

Chapter 6

Conclusion

This research aims to develop a renormalization group method to tackle the problem of reducing the number of degrees of freedom necessary to describe fluid turbulence. In this study, we restrict our attention to homogeneous, isotropic and stationary turbulence in an incompressible fluid. A brief summary of the theory of fluid turbulence is given in chapter 1.

In chapter 2, we review renormalization group methods applied to fluid turbulence: field-theoretic and recursive-type renormalization group method. The approach of each type of renormalization group method is discussed with critiques. In addition to discussions of the field-theoretic renormalization group method, we briefly review the method of Martin, Siggia and Rose [35] in appendix A. The method developed in this thesis is one of the recursive-type renormalization group methods.

In chapter 3, we have reformulated the iterative-averaging renormalization group method, originally introduced by McComb [49] and developed in series of papers [51, 56, 58, 63]. We begin by re-writing the Navier-Stokes equation on the interval $0 \leq k \leq k_0$, which has the form:

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha(\boldsymbol{k},t) = \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t) + M_{\alpha\beta\gamma}(\boldsymbol{k}) \int_{j<k_0} d^3j \; u_\beta(\boldsymbol{j},t) u_\gamma(\boldsymbol{k}-\boldsymbol{j},t), \quad (6.1)$$

where the maximum cut-off wave-number, k_0, and the stirring force, $\mathcal{F}^{\ll}$, are

defined in section 3.2. Especially, it should be marked here that the specifications of our stirring force appeared in equation (6.1) is different from those of the multivariate normal model described in section 1.2.

The application of renormalization group method applied to fluid turbulence in general involves the elimination of high wave-number modes (i.e. small eddies in real space) and the supplement of their effects to the kinematic viscosity; which has been referred to as the shell elimination. In order to eliminate the first shell of equation (6.1), we employ the techniques in section 3.3 and 3.4, which are:

conditional average evaluated over our biased sub-ensemble to be chosen as the sub-set of realizations for which the low wave-number modes differ from $\boldsymbol{u}^-$ by a small amount: ξ in figure 3.1;

perturbation expansion for the eliminated high-wave-number modes of the velocity field, $\boldsymbol{u}^+$, based on a new model of fluid turbulence in which the mode-mode couplings across the cut-off wave-number are suppressed;

approximations made based on physical or mathematical grounds in terms of previously accumulated knowledge about the properties of fluid turbulence: one is a closure approximation and the other is a Markovian approximation.

As described in chapter 2, our approach is different from that of field-theoretic renormalization group method based on the multivariate normal model; and thus it is necessary to introduce the conditional average to replace the original filtered average. The conditional average has been developed [49, 63], and the numerical investigation of probing the basic applicability of using this technique is currently being carried out [59, 77]. Due to the nature of the Navier-Stokes equation, however, even if we succeed in holding the low wave-number modes of the velocity field, $\boldsymbol{u}^-$, approximately constant under the conditional average, the conditionally averaged properties of the high wave-number modes of the velocity field, $\boldsymbol{u}^+$, still remain unknown (see section 3.3). Therefore, the method

requires a representative model for which the conditional average could be exactly evaluated once. The first attempt was made by McComb and Watt [51], who chose the model as a first order Taylor expansion of $\boldsymbol{u}^+$ at the maximum cut-off wave-number: $k = k_0$. As previously discussed, the use of a Taylor series to represent the chaotic velocity field has been criticized.

Alternatively, in chapter 3, we suggest a new model which provides the possibility of describing the high and low wave-number modes independently, by suppressing the phase information between them. The model field is defined as equation (3.27) in section 3.3.2, which has the form:

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] v_\alpha^+(\boldsymbol{k},t)$$

$$= M^+_{\alpha\beta\gamma}(\boldsymbol{k}) \int^+ d^3j \left[v_\beta^-(\boldsymbol{j},t) v_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t) + v_\beta^+(\boldsymbol{j},t) v_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t)\right]. \quad (6.2)$$

The ansatz for our model field in which the couplings between the high and low wave-number modes are suppressed, is based on the localness of the mode-mode coupling interaction in wave-number space. As pointed out in section 3.5, however, a more rigorous assessment of the validity of such a model for the Navier-Stokes turbulence remains to be done, and this is a current topic of our studies.

In section 3.4, a formal perturbation expansion is then made on the basis of the new model field. As a result, we have the systematic derivation of a modified Navier-Stokes equation for the remaining low wave-number modes on the interval $0 \le k \le k_1$, and an expression for the effective viscosity:

$$\left[\frac{\partial}{\partial t} + \nu_1(k)k^2\right] u_\alpha^-(\boldsymbol{k},t) = \mathcal{F}_\alpha^{\ll}(\boldsymbol{k},t) + M^-_{\alpha\beta\gamma}(\boldsymbol{k}) \int^- d^3j \; u_\beta^-(\boldsymbol{j},t) u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t), \quad (6.3)$$

where

$$\nu_1(k) = \nu_0 + \delta\nu_0(k) \quad (6.4)$$

with

$$\delta\nu_0(k) = \frac{1}{k^2} \int^+ d^3j \; \frac{2\mathrm{Tr}[-M^-_{\alpha\beta\gamma}(\boldsymbol{k}) M^+_{\gamma\rho\sigma}(\boldsymbol{k}-\boldsymbol{j}) D_{\beta\sigma}(\boldsymbol{j})]}{\nu_0 j^2 + \nu_0 |\boldsymbol{k}-\boldsymbol{j}|^2} Q_v^+(j). \quad (6.5)$$

We have derived the effective viscosity representing the effect of the eliminated (or subgrid) modes upon the modified Navier-Stokes equation for the remaining modes, and this is in effect a *subgrid model* for large-eddy simulations.

In section 4.2, we then expand the covariance of the velocity field rather than the velocity field itself to relate the phase and amplitude difference between our model field and the velocity field in Fourier space. This allows us to extend our results [equations (6.3) - (6.5)] to further shells, and the following calculations of the theory are presented. When the fixed point for the effective viscosity is numerically calculated, a value for the Kolmogorov constant is found (see figure 4.2). At first order, this is the same result as in the previous report [51], but the new procedure allows us to investigate higher orders.

The other approximations made in our theory are the so-called boundary-layer-type approximations in section 3.4.3. Those approximations have been discussed previously [51], but our systematic procedure allows us to give further attention. The first approximation is related to the closure problem of turbulence, where the triple moment $\langle \boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(1)}\rangle_c$ has been dropped in comparison to $\boldsymbol{v}^-\langle \boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(0)}\rangle_c$. It should be noted that this approximation is made for convenience rather than necessity; because the term $\langle \boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(1)}\rangle_c$ can be evaluated by substituting the solution of $\boldsymbol{u}^{+(1)}$ given in section 3.4.1, and it can be found that the process will produce $\langle \boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(0)}\rangle_c\boldsymbol{u}^-$ and others of order of $\mathcal{E}\{\langle \boldsymbol{\Phi}^-\boldsymbol{\Phi}^-\rangle_c\}$. In this consideration, our closure approximation is different from that associated with the multivariate normal model (see section 1.2). Although, in this thesis, we truncate the series by ignoring the triple $\boldsymbol{u}^{+(0)}$ moment as a small amount, it may be interesting to see how the higher-order terms linear in $\boldsymbol{u}^-$ change the current result, and this will be the subject of future work.

The other approximation in section 3.4.3 is the well-known Markovian approximation, where the high wave-number modes are taken to evolve rapidly compared to the low wave-number modes. However, the introduction of an evolution equa-

tion for $\boldsymbol{u}^{+(0)}\boldsymbol{u}^{+(1)}$ in the method has been doubtful, while the straightforward substitution for $\boldsymbol{u}^{+(1)}$ could be possible (see reference [50] for instance). In this respect, in section 4.3, we show that our alternative procedure should be carried out, in order to make the proper Markovian approximation.

In addition to the discussion of our iterative-averaging renormalization group method, in section 4.4, we discuss the treatment of the triple non-linear contributions in our theory. As previously mentioned, the iterative-averaging renormalization group method developed over the years [49, 51, 56, 58, 63] has the result that an equation for the remaining modes does not contain a term $\boldsymbol{u}^{-}\boldsymbol{u}^{-}\boldsymbol{u}^{-}$, which breaks the form invariance of the equation of motion under renormalization group transformation. However, there was an inadequate explanation of this respect in the procedure of their theories. In section 4.4, we show that all the triple non-linear contributions are of order of $\mathcal{E}\{\langle\boldsymbol{\Phi}^{-}\boldsymbol{\Phi}^{-}\rangle_c\}$ except a term which gives a viscosity increment. As mentioned in section 4.4, however, a numerical investigation of the uncertainty: $\boldsymbol{\Phi}^{-}$ should be carried out in the future.

Finally, the method developed in chapter 3 is extended to the problem of passive scalar transport in chapter 5. A relevant model field analogous to that for the velocity field is suggested, and a double expansion for the passive scalar and the velocity field is applied. As a result, our procedures give a systematic derivation of the effective diffusivity representing the (conditional) mean effects of the eliminated modes upon the retained modes of the passive scalar field.

Appendix A

An Interpretation of Martin, Siggia and Rose Method

A.1 Introductory remarks

The functional integral formalism for the description of the classical statistical dynamics, based on the work of Martin, Siggia and Rose [35] (referred to as MSR, hereafter), is reviewed. We follow the representation of Jensen [78], who extensively summarized the literature of the application of the MSR method.

In addition to discussions about renormalized perturbation theory in section 1.2.4 and field-theoretic renormalization group method in section 2.2, an application of the MSR method to a classical system governed by the Navier-Stokes equation is briefly considered in the last section.

A.2 Classical statistical dynamics

A.2.1 Stochastic differential equations

Consider the class of stochastic differential equations which can be written in the following form:

$$\partial_{t_1}\psi(1) = U_1(1) + U_2(12)\psi(2) + U_3(123)\psi(2)\psi(3) + \delta(t_1 - t_0)\psi_0(\mathbf{1}), \quad \text{(A.1)}$$

where the meanings of notations are:

- $\psi(1)$ is in general a real-valued classical field which has a discontinuity at $t_1 = t_0$: $\psi(1) \equiv \theta(t_1 - t_0)\psi(1)$;
- The index "1" represents the time and all other variables and the bold index "**1**" stands for all other variables without the time: $1 \equiv (\mathbf{1}, t_1)$;
- The duplication of index implies a summation for internal coordinate;
- The forces and interactions $U_i(1 \ldots i) = \bar{U}_i(1 \ldots i) + \tilde{U}_i(1 \ldots i)$ can be decomposed into a deterministic piece $\bar{U}_i(1 \ldots i)$ and a random piece $\tilde{U}_i(1 \ldots i)$ with known statistics. Here, *deterministic* means $\bar{U}_i(1 \ldots i)$ is local and stationary, thus

$$\bar{U}_i(1 \ldots i) \sim \delta(1-2) \ldots \delta(1-i); \tag{A.2}$$

- Finally, the initial condition generally consists of a deterministic and a random piece: $\psi_0 = \bar{\psi}_0 + \tilde{\psi}_0$.

Then, the fundamental quantities of the classical system governed by equation (A.1) are:

- *The mean field*:

$$\langle \psi(1) \rangle, \tag{A.3}$$

  where the bracket $\langle \cdots \rangle$ is used to indicate averages over the random forces, interactions or initial conditions;

- *The correlation function*:

$$Q(12) \equiv \langle \delta\psi(1)\delta\psi(2) \rangle = \langle\langle \psi(1)\psi(2) \rangle\rangle, \tag{A.4}$$

  where $\langle\langle \cdots \rangle\rangle$ denotes the cumulant;

- *The averaged response function* to infinitesimal external perturbations:

$$R(12) \equiv \langle \delta\psi(1)/\delta\bar{U}_1(2)\rangle_{\bar{U}_1(2)=0}. \tag{A.5}$$

The basic and major problem in both classical statistical dynamics and quantum mechanics is the calculation of these fundamental quantities of the system.

A.2.2 Randomly stirred Navier-Stokes turbulence

An important case of (A.1) is the Navier-Stokes equation for a randomly stirred, incompressible fluid:

$$\partial_t \boldsymbol{u} + \boldsymbol{D} : (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \nu_0 \boldsymbol{\nabla}^2 \boldsymbol{u} + \boldsymbol{f} \tag{A.6}$$

where $\boldsymbol{D}$ is the transverse projection operator, ν_0 is the kinematic viscosity and $\boldsymbol{f}$ is a transverse, Gaussian random stirring force.

This is an example of a problem with a linear random force. The correspondence with equation (A.1) follows by identifying:

$$\psi(1) = u_\alpha(\boldsymbol{x}_1, t_1)\theta(t_1 - t_2); \tag{A.7}$$

$$\tilde{U}_1(1) = f_\alpha(\boldsymbol{x}_1, t_1); \tag{A.8}$$

$$\bar{U}_2(12) = \nu_0 \boldsymbol{\nabla}^2 \, \delta(1-2); \tag{A.9}$$

$$\bar{U}_3(123) = -\boldsymbol{D} : \boldsymbol{\nabla}\cdot \, \delta(1-2)\delta(1-3). \tag{A.10}$$

A.3 Basic idea of Martin, Siggia and Rose

In order to describe the statistical properties of a classical dynamic system which is governed by a stochastic differential equation, we need a theory for the calculation of correlation (fluctuation) functions and response functions (averaged Green's functions). Unfortunately, the evolution of these second-order moment functions depends in general upon higher-order correlation functions due to the

non-linear interactions. Therefore, the resulting hierarchy of equations can only be closed by some truncation procedure.

In 1973, Martin, Siggia and Rose [35] developed a method for overcoming this difficulty. They succeed in deriving closed, self-consistent equations for the evolution of the statistical correlation and response functions by applying techniques in the study of interacting quantum fields to statistical dynamics of classical systems. In order to take advantage of the methods of quantum field theory, MSR treat the classical field $\psi(1)$ as a Heisenberg operator. The classical correlation functions are then defined to be vacuum expectation values of time-ordered products of these operators e.g.,

$$Q(12) \equiv \langle\langle \psi(1)\psi(2)\rangle\rangle_{+}, \tag{A.11}$$

where the subscript "+" indicates the time ordering operator, which has the definition

$$\langle\langle \psi(1)\psi(2)\rangle\rangle_{+} = \begin{cases} \langle\langle \psi(1)\psi(2)\rangle\rangle & \text{if} \quad t_1 > t_2 \\ \langle\langle \psi(2)\psi(1)\rangle\rangle & \text{if} \quad t_2 > t_1. \end{cases} \tag{A.12}$$

The important contribution of MSR was the introduction of an adjoint operator $\hat{\psi}(\mathbf{1}, t)$ which does not commute with $\psi(\mathbf{2}, t)$:

$$\left[\psi(\mathbf{1}, t), \hat{\psi}(\mathbf{2}, t)\right] = \delta(\mathbf{1} - \mathbf{2}). \tag{A.13}$$

The averaged response function is then given by the time-ordered expectation value of $\psi(1)$ and $\hat{\psi}(2)$, thus:

$$R(12) = \langle\langle \psi(1)\hat{\psi}(2)\rangle\rangle_{+}, \tag{A.14}$$

which is the Green's function in the theory of quantum fields. Note the difference between equation (A.11) for $Q(12)$ and equation (A.14) for $R(12)$ due to a new appearance of $\hat{\psi}$.

Moreover, the adjoint operator, $\hat{\psi}(1)$ makes it possible to construct a Hamiltonian which generates the equation of motions for the operators $\psi(1)$ and $\hat{\psi}(1)$.

However, it should be noted here that this approach is only applicable to the statistical dynamics governed by equation (A.1), when forces and interactions are deterministic. With this limitation, we may write the Hamiltonian H as

$$H \equiv \hat{\psi}(1)\partial_{t_1}\psi(1) \tag{A.15}$$

$$= \hat{\psi}(1)[\bar{U}_1(1) + \bar{U}_2(12)\psi(2) + \bar{U}_3(123)\psi(2)\psi(3)] \tag{A.16}$$

and such that the equation of motions can be derived with the commutation relation, (A.13), which are:

$$\partial_{t_1}\psi(1) \equiv \left[\psi(1), H\right] \tag{A.17}$$

$$= \bar{U}_1(1) + \bar{U}_2(12)\psi(2) + \bar{U}_3(123)\psi(2)\psi(3); \tag{A.18}$$

$$\partial_{t_1}\hat{\psi}(1) \equiv \left[\hat{\psi}(1), H\right] \tag{A.19}$$

$$= -\bar{U}_2(21)\hat{\psi}(2) - \bar{U}_3(231)\hat{\psi}(2)\psi(3). \tag{A.20}$$

Note that equation (A.18) is just equation (A.1) again. Here, equations (A.15), (A.17) and (A.19) are identical forms which appear in quantum field theory. Therefore, under the circumstances, it is now possible to apply the techniques of quantum field theory to calculate the statistics of correlation functions and response functions defined by equations (A.11) and (A.14), respectively.

The original work of MSR [35] presented the method parallel to the mathematical formalism of quantum field theory and the statistical dynamics of a classical stochastic field for deterministic forces and interactions; in fact, the MSR approach is identical with the classical version of Schwinger's field theory [79] or the operator formulation [80] in the language of quantum field theory. MSR also considered a system stirred by a random force $\tilde{U}_1(1)$ with Gaussian statistics providing that $\tilde{U}_1(1)$ generates an additional Hamiltonian, which has the form of

$$\hat{\psi}(1)\langle\langle\tilde{U}_1(1)\tilde{U}_1(2)\rangle\rangle\hat{\psi}(2). \tag{A.21}$$

Later, Deker and Haake [81] and Phythian [82] extended the MSR method to

multiplicative random forces and Deker [83] also modified and extended it to non-Gaussian initial conditions.

A.4 Functional integral formalism

As the approach of Feynman's path integral formalism [84] has been proven fruitful in quantum mechanics, It might be also helpful to recast the operator formalism of MSR in terms of a functional integral formalism. In fact, the functional formalism has been consistently applied to describe classical systems in various ways [3]. In this section, the previous attempts to apply the functional formulation are briefly reviewed, and advantages of reformulating the MSR method in terms of a functional integral formalism are discussed.

A.4.1 Functional formalism for classical systems

Almost five decades ago, the first functional formulation of classical systems, especially for the turbulence problem was introduced by Hopf [85]. He considered the functional probability distribution $P\{\psi(\mathbf{1}), t_1\}$ which contains a complete statistical description of the classical dynamical system, and derived its governing equation from the Navier-Stokes equation. Later, Lewis and Kraichnan [86] obtained the equation for the evolution in time of the more general functional $P\{\psi(1)\}$. Since the basic concern of MSR was that the classical field has the value $\psi(1)$, this section is based on the work of Lewis and Kraichnan.

Without going into great detail here, it may be worth highlighting the property of the characteristic functional defined by Lewis and Kraichnan [86]. The characteristic functional $M\{\eta\}$ of the functional probability distribution $P\{\psi\}$ can take the form

$$M\{\eta\} = \int D[\psi]\ P\{\psi\} \exp[\psi(1)\eta(1)], \tag{A.22}$$

where the notation

$$\int D[\psi] \cdots \tag{A.23}$$

represents a functional integral: the integration is taken over all functions, ψ. In general, the main property of the characteristic functional is that all orders of correlation functions are given by functional derivatives of $M\{\eta\}$ with respect to η. Thus, the general nth-order correlation can be obtained from equation (A.22). This takes the forms

$$\langle \psi(1) \dots \psi(n) \rangle = \left. \frac{\delta^n M\{\eta\}}{\delta\eta(1) \dots \delta\eta(n)} \right|_{\eta=0}, \tag{A.24}$$

where $\delta/\delta\eta$ stands for the functional derivative.

As a consequence of MSR introducing the adjoint operator $\hat{\psi}$ (and the non-classical algebra), the previous characteristic functional of (A.22) can be extended to the much more useful form:

$$M\{\eta, \hat{\eta}\} = \int D[\psi] \int D[\hat{\psi}] \; P\{\psi, \hat{\psi}\} \exp[\psi(1)\eta(1) + \hat{\psi}(1)\hat{\eta}(1)]. \tag{A.25}$$

Thus, it is now possible to derive *not only* correlation functions *but also* Green's functions including the averaged response function from equation (A.25). Undoubtedly, this is the main contribution of the work of MSR. The further details of the functional formalism for the MSR method are discussed in the following section.

A.4.2 The generalized MSR method

An alternative approach of a functional integral formalism to the description of the MSR method was first introduced by Janssen [87] and DeDominicis [88]. Whereas a second set of operator $\hat{\psi}$ was treated as an anti-commuted adjoint operator in the work of MSR, a new field $\hat{\psi}$ occurs naturally as a conjugate momentum variable in the functional integral formalism, closely related to Feynman's path integral formulation for quantum mechanics. Furthermore, this approach

has been proven to be applicable to a wider range of physical problems. Phythian investigated the functional integral formalism further [89] and showed that the statistical equations of motion for multiplicative random force can be derived with this approach [90]. Later, Jensen [78] summarized the functional integral approach for various problems.

Now, consider a generating functional previously shown in equation (A.25):

$$M\{\eta,\hat{\eta}\} \equiv \int D[\psi] \int D[\hat{\psi}] P\{\psi,\hat{\psi}\} \exp[\psi(1)\eta(1) + \hat{\psi}(1)\hat{\eta}(1)], \tag{A.26}$$

where the formulation, using the language of generating functional and so on, is due to Schwinger [79]. In quantum-mechanical rule [84], the probability density functional $P\{\psi,\hat{\psi}\}$ in equation (A.25) has the form with the Lagrangian L for the system,

$$P\{\psi,\hat{\psi}\} = \langle \exp(-L) \rangle. \tag{A.27}$$

Accordingly, performing the average in equation (A.27) gives the definitions of an averaged Lagrangian $\mathcal{L}$ and an averaged Hamiltonian $\mathcal{H}$

$$\langle \exp(-L) \rangle \equiv \exp(-\mathcal{L}) \equiv \exp\{-[\hat{\psi}(1)\partial_{t_1}\psi(1) - \mathcal{H}]\} \tag{A.28}$$

with

$$\mathcal{H} = \bar{\mathcal{H}} + \tilde{\mathcal{H}}, \tag{A.29}$$

$$\begin{aligned} \bar{\mathcal{H}} = {} & \hat{\psi}(1)[\bar{U}_1(1) + \bar{U}_2(12)\psi(2) + \bar{U}_3(123)\psi(2)\psi(3) \\ & + \delta(t_1 - t_0)\bar{\psi}_0(\mathbf{1})], \end{aligned} \tag{A.30}$$

$$\begin{aligned} \tilde{\mathcal{H}} = {} & \ln \Big\langle \exp\{\hat{\psi}(1)[\tilde{U}_1(1) + \tilde{U}_2(12)\psi(2) + \tilde{U}_3(123)\psi(2)\psi(3) \\ & + \delta(t_1 - t_0)\tilde{\psi}_0(\mathbf{1})]\}\Big\rangle, \end{aligned} \tag{A.31}$$

It is now clear to see that the definition of equation (A.29) has the advanced and practical form in comparison with equation (A.15); reminding that equation (A.15) is just equation (A.30).

Once the generating functional is constructed for the systems, the statistical average of any functional F of ψ and $\hat{\psi}$ can be readily defined as

$$\left\langle F\{\psi,\hat{\psi}\}\right\rangle = F\left\{\frac{\delta}{\delta\eta},\frac{\delta}{\delta\hat{\eta}}\right\}M\{\eta,\hat{\eta}\}. \tag{A.32}$$

It is also convenient to define a cumulant generating functional as,

$$Z\{\eta,\hat{\eta}\} = \ln\left[M\{\eta,\hat{\eta}\}\right]. \tag{A.33}$$

Then, the nth-order of cumulant for ψ and $\hat{\psi}$ is

$$\begin{aligned}&\langle\langle\psi(1)\ldots\psi(i)\hat{\psi}(i+1)\ldots\hat{\psi}(n)\rangle\rangle\\ &= \left.\frac{\delta^n Z\{\eta,\hat{\eta}\}}{\delta\eta(1)\ldots\delta\eta(i)\delta\hat{\eta}(i+1)\ldots\delta\hat{\eta}(n)}\right|_{\eta=0,\ \hat{\eta}=0}.\end{aligned} \tag{A.34}$$

In general, the functional integral representation for the cumulant generating functional Z, equation (A.33), is too complicated for practical calculation of statistical quantities. However, the equations for statistical correlation and response functions can be obtained rather easily. Since the cumulant generating functional Z is a functional of η and $\hat{\eta}$, we can have trivial identities:

$$\frac{\delta}{\delta\hat{\psi}}Z\{\eta,\hat{\eta}\} = 0; \tag{A.35}$$

$$\frac{\delta}{\delta\psi}Z\{\eta,\hat{\eta}\} = 0. \tag{A.36}$$

Upon using equations (A.26), (A.27), (A.28) and (A.33), we can rewrite equation (A.35) as

$$\begin{aligned}\frac{\delta}{\delta\hat{\psi}(1)}Z\{\eta,\hat{\eta}\} &= \frac{1}{M\{\eta,\hat{\eta}\}}\frac{\delta}{\delta\hat{\psi}(1)}M\{\eta,\hat{\eta}\}\\ &= \frac{1}{M\{\eta,\hat{\eta}\}}\left[-\partial_{t_1}\psi(1)+\frac{\delta}{\delta\hat{\psi}(1)}\mathcal{H}+\hat{\eta}(1)\right]\\ &= 0.\end{aligned} \tag{A.37}$$

With the definition of average, equation (A.32), we now have the equation of motion for $\langle\psi\rangle$ as

$$\frac{\partial_{t_1}\langle\psi(1)\rangle}{M\{\eta,\hat{\eta}\}} - \frac{1}{M\{\eta,\hat{\eta}\}}\left\langle\frac{\delta}{\delta\hat{\psi}(1)}\mathcal{H}\right\rangle - \hat{\eta}(1) = 0. \tag{A.38}$$

Similarly, the equation of motion for $\langle\hat{\psi}\rangle$ can be derived from equation (A.36), which has the form

$$\frac{\partial_{t_1}\langle\hat{\psi}(1)\rangle}{M\{\eta,\hat{\eta}\}} + \frac{1}{M\{\eta,\hat{\eta}\}}\left\langle\frac{\delta}{\delta\psi(1)}\mathcal{H}\right\rangle + \eta(1) = 0. \tag{A.39}$$

Now, equations (A.38) and (A.39) provide the identical results of MSR, when $\mathcal{H}$ only has deterministic forces and interactions, $\bar{\mathcal{H}}$ [i.e. equation (A.30)]:

$$\partial_{t_1}\langle\psi(1)\rangle = \bar{U}_1(1) + \bar{U}_2(12)\langle\psi(2)\rangle + \bar{U}_3(123)\langle\psi(2)\psi(3)\rangle; \tag{A.40}$$

$$\partial_{t_1}\langle\hat{\psi}(1)\rangle = -\bar{U}_2(21)\langle\hat{\psi}(2)\rangle - 2\bar{U}_3(231)\langle\hat{\psi}(2)\psi(3)\rangle, \tag{A.41}$$

which are averaged equations for (A.18) and (A.20). Here, we set

$$\eta = \hat{\eta} = 0, \tag{A.42}$$

and normalize the generating functional as

$$M\{\eta,\hat{\eta}\}\Big|_{\eta=0,\ \hat{\eta}=0} = 1. \tag{A.43}$$

Furthermore, the Dyson equations for correlation and averaged response functions follow by functional differentiating equation (A.38) with respect to $\eta(1')$ and $\hat{\eta}(1')$:

$$\begin{aligned}&\frac{\partial_{t_1}\langle\langle\psi(1)\psi(1')\rangle\rangle}{M^2\{\eta,\hat{\eta}\}}\\ &-\left[\frac{1}{M\{\eta,\hat{\eta}\}}\left\langle\psi(1')\frac{\delta}{\delta\hat{\psi}(1)}\mathcal{H}\right\rangle - \frac{1}{M^2\{\eta,\hat{\eta}\}}\langle\psi(1')\rangle\left\langle\frac{\delta}{\delta\hat{\psi}(1)}\mathcal{H}\right\rangle\right]\\ &= 0;\end{aligned} \tag{A.44}$$

$$\begin{aligned}&\frac{\partial_{t_1}\langle\langle\psi(1)\hat{\psi}(1')\rangle\rangle}{M^2\{\eta,\hat{\eta}\}}\\ &-\left[\frac{1}{M\{\eta,\hat{\eta}\}}\left\langle\hat{\psi}(1')\frac{\delta}{\delta\psi(1)}\mathcal{H}\right\rangle - \frac{1}{M^2\{\eta,\hat{\eta}\}}\langle\hat{\psi}(1')\rangle\left\langle\frac{\delta}{\delta\psi(1)}\mathcal{H}\right\rangle\right]\\ &= \delta(1-2).\end{aligned} \tag{A.45}$$

Formally, equations (A.38), (A.44) and (A.45) represent a complete description of the statistical dynamics of classical systems which are governed by stochastic

differential equations defined by equation (A.1). Although these equations prove in general to be too complicated to solve directly, the advantage of such a description is to serve a starting point for several different systematic perturbation schemes. In the next section, this point of view will be discussed for the case of Navier-Stokes turbulence with a random stirring force.

A.5 Application of MSR method to turbulence

As an illustration of the MSR method, consider Navier-Stokes Turbulence. Upon using notations (A.7)- (A.10), the Navier-Stokes equation for a randomly stirred, incompressible fluid described in equation (A.6) can be written in the form

$$\partial_{t_1}\psi(1) = \tilde{U}_1(1) + \bar{U}_2(12)\psi(2) + \bar{U}_3(123)\psi(2)\psi(3), \tag{A.46}$$

where $\tilde{U}_1(1)$ is Gaussian random force. Then, an averaged Hamiltonian for Navier-Stokes turbulence can be expressed as follows [see equations (A.30) and (A.31)]:

$$\begin{aligned}\mathcal{H} \equiv\ & \hat{\psi}(1)[\bar{U}_2(12)\psi(2) + \bar{U}_3(123)\psi(2)\psi(3)] \\ & + \frac{1}{2}\hat{\psi}(1)\langle\langle\tilde{U}_1(1)\tilde{U}_1(2)\rangle\rangle\hat{\psi}(2),\end{aligned} \tag{A.47}$$

where we have expanded out the cumulant function $\langle\langle\tilde{U}_1(1)\tilde{U}_1(2)\rangle\rangle$ in terms of the cumulant average of the Gaussian random force.

A.5.1 Renormalized perturbation theory

It has been previously shown that there are no restrictions to deterministic or random interactions in the functional integral formalism. Therefore, by substituting equation (A.47) into equations (A.44) and (A.45), the statistical equations of motion for correlation and response functions can be obtained as:

$$\partial_{t_1} Q(11') - \bar{U}_2(12)Q(21')$$

$$-\bar{U}_3(123)\langle\langle\psi(2)\psi(3)\psi(1')\rangle\rangle = \frac{1}{2}R(1'2)\langle\langle\tilde{U}_1(1)\tilde{U}_1(2)\rangle\rangle; \qquad \text{(A.48)}$$

$$\partial_{t_1} R(11') - \bar{U}_2(12)R(21')$$

$$-\bar{U}_3(123)\langle\langle\psi(2)\psi(3)\hat{\psi}(1')\rangle\rangle = \delta(1-1'), \qquad \text{(A.49)}$$

where $Q(12)$ for $\langle\langle\psi(1)\psi(2)\rangle\rangle$ and $R(12)$ for $\langle\langle\psi(1)\hat{\psi}(2)\rangle\rangle$.

Although the MSR approach presents a complete description of the statistical dynamics of classical systems and provides several different systematic perturbation schemes, at least for the case of Navier-Stokes turbulence with a random stirring force, it has been unable to investigate the problem no further than the lowest-order renormalized perturbation theory (direct interaction approximation: DIA) due to its complications. Nevertheless, in DIA, we have closed forms of equation (A.48) and (A.49):

$$\partial_{t_1} Q(11') - \bar{U}_2(12)Q(21')$$

$$-\bar{U}_3(123)\bar{U}_3(456)R(24)Q(35)Q(1'6) - \bar{U}_3(123)\bar{U}_3(456)Q(24)R(35)Q(1'6)$$

$$= \frac{1}{2}R(1'2)\langle\langle\tilde{U}_1(1)\tilde{U}_1(2)\rangle\rangle; \qquad \text{(A.50)}$$

$$\partial_{t_1} R(11') - \bar{U}_2(12)R(21')$$

$$-\bar{U}_3(123)\bar{U}_3(456)R(24)Q(35)R(1'6) - \bar{U}_3(123)\bar{U}_3(456)Q(24)R(35)R(1'6)$$

$$= \delta(1-1'). \qquad \text{(A.51)}$$

A.5.2 Renormalization group theory

Since Forster, Nelson and Stephen [40, 41] used RG formalism in a way of solving the problems that arise due to non-linear coupling for the stirred hydrodynamics,

the functional integral formalism of MSR has been applied to explain a theory of turbulence in terms of RG approach [42, 44, 45].

The starting point of RG calculations based on the MSR formalism can be writing $\mathcal{H}$ defined in equation (A.47) as

$$\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{H}_1, \tag{A.52}$$

where

$$\mathcal{H}_0 = \hat{\psi}(1)\bar{U}_2(12)\psi(2) + \frac{1}{2}\hat{\psi}(1)\langle\langle \tilde{U}_1(1)\tilde{U}_1(2)\rangle\rangle\hat{\psi}(2), \tag{A.53}$$

$$\mathcal{H}_1 = \hat{\psi}(1)\bar{U}_3(123)\psi(2)\psi(3). \tag{A.54}$$

By substituting $\mathcal{H}_0$ into equations (A.44) and (A.45), the correlation function and the response function have the forms:

$$Q_0(12) = \frac{1/2\ \langle\langle \tilde{U}_1(1)\tilde{U}_1(2)\rangle\rangle\delta(1-1')}{[\partial_{t_1}\delta(t_1-t_2) - \bar{U}_2(12)]^2}; \tag{A.55}$$

$$R_0(12) = \frac{\delta(1-1')}{\partial_{t_1}\delta(t_1-t_2) - \bar{U}_2(12)}, \tag{A.56}$$

where the subscript "0" denotes the zeroth-order theory. In the theory of quantum field, equations (A.55) and (A.56) are expressed in terms of bare propagators.

In principle, having got the Wyld-Feynman rules, it is now possible to calculate the any order of Q and R in perturbation theory. However, it has been found to be difficult to implement in practice, due to the problem arising from the strong mode-mode coupling of the Navier-Stokes velocity field. The relevant discussions can be found in section 2.2. Therefore, it is as yet questionable that this type of RG method (referred to as the field-theoretic RG method in chapter 2) can play a useful role in the fixed point dynamics in the study of turbulence, and support phenomenological subgrid models for large-eddy simulations, unless the new development of suitable approximation methods.

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